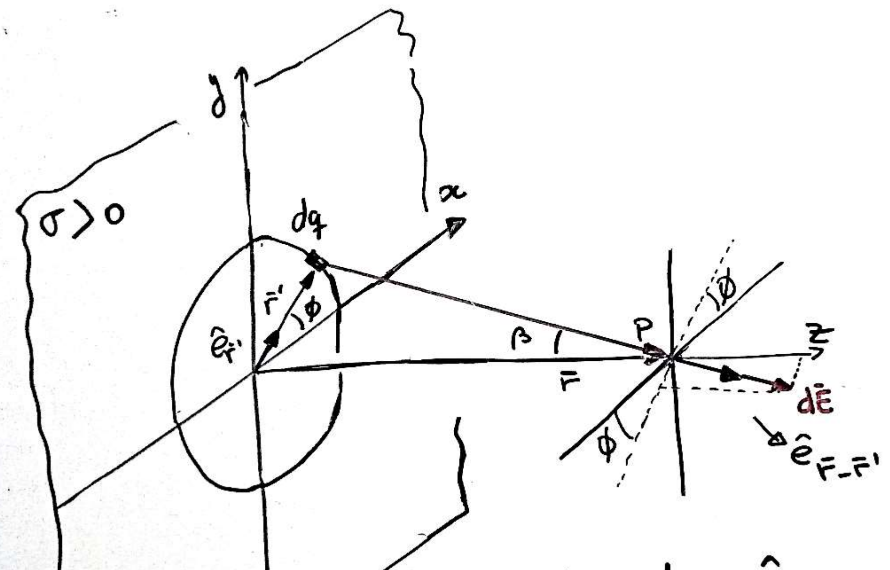
CALCULO DE $\bar{E}$ PARA UN PLANO INFINITO



$$d\bar{E}(P) = k_e \frac{dq}{|\bar{r}-\bar{r}'|^2}\,\hat{e}_{\bar{r}-\bar{r}'}$$

$$|\bar{r}'| = r' \qquad dq = \sigma\, ds = \sigma\, r'\, dr'\, d\phi$$

$$\bar{r}' = r'\cos\phi\,\hat{e}_x + r'\,\mathrm{sen}\,\phi\,\hat{e}_y$$

$$\bar{r} = z\,\hat{e}_z$$

$$\bar{r}-\bar{r}' = z\,\hat{e}_z - r'\cos\phi\,\hat{e}_x - r'\,\mathrm{sen}\,\phi\,\hat{e}_y$$

$$|\bar{r}-\bar{r}'|^2 = z^2 + r'^2\cos^2\phi + r'^2\,\mathrm{sen}^2\phi = z^2 + r'^2$$

$$\hat{e}_{\bar{r}-\bar{r}'} = \cos\beta\,\hat{e}_z - \mathrm{sen}\beta\cos\phi\,\hat{e}_x - \mathrm{sen}\beta\,\mathrm{sen}\phi\,\hat{e}_y$$

$$d\bar{E}(P) = k_e \frac{\sigma\, r'\, dr'\, d\phi}{(z^2+r'^2)}\left[\cos\beta\,\hat{e}_z - \mathrm{sen}\beta\cos\phi\,\hat{e}_x - \mathrm{sen}\beta\,\mathrm{sen}\,\phi\,\hat{e}_y\right]$$

Tener en cuenta que,

$$\cos\beta = \frac{z}{(r'^2+z^2)^{1/2}} \qquad \mathrm{sen}\,\beta = \frac{r'}{(r'^2+z^2)^{1/2}}$$

Para encontrar el campo $\bar{E}$ total hay que integrar en todo el plano

$$\bar{E}(P) = k_e\sigma\left[-\int_0^\infty\int_0^{2\pi} \frac{r'^2\cos\phi\, d\phi\, dr'\,\hat{e}_x}{(r'^2+z^2)^{3/2}} - \int_0^\infty\int_0^{2\pi}\frac{r'^2\,\mathrm{sen}\,\phi\, d\phi\, dr'\,\hat{e}_y}{(r'^2+z^2)^{3/2}} + \int_0^\infty\int_0^{2\pi}\frac{z\, r'}{(r'^2+z^2)^{3/2}}\, d\phi\, dr'\,\hat{e}_z\right]$$

Como $\int_0^{2\pi}\cos\phi\, d\phi = 0$ y $\int_0^{2\pi}\mathrm{sen}\,\phi\, d\phi = 0$

$$\Rightarrow \bar{E}(P) = k_e\sigma z\, 2\pi \int_0^\infty \frac{r'}{(z^2+r'^2)^{3/2}}\, dr'\,\hat{e}_z$$

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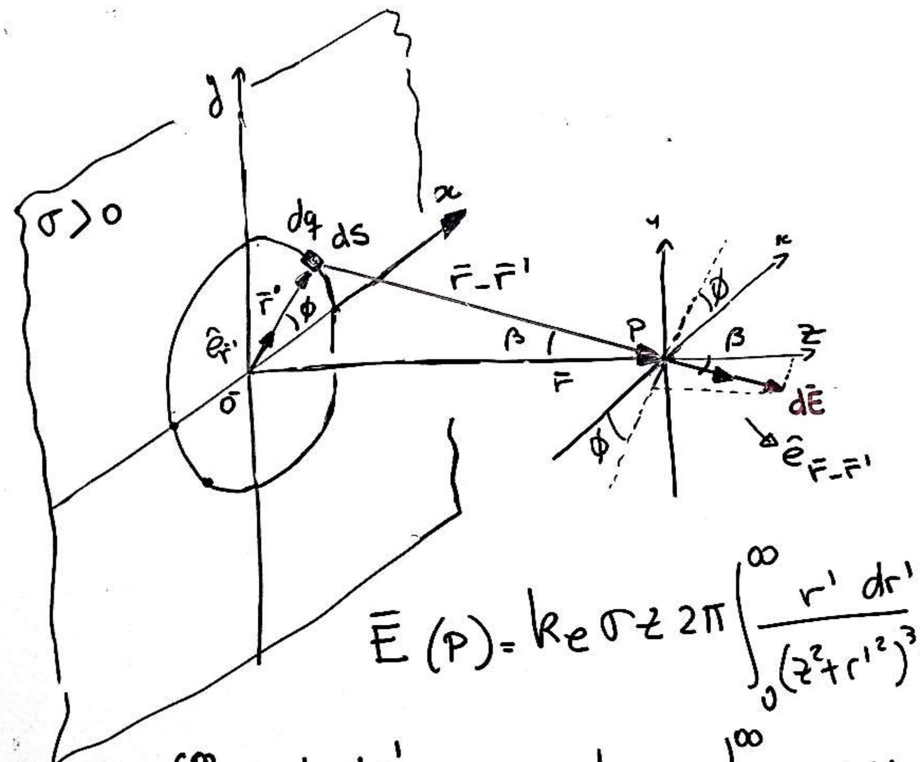


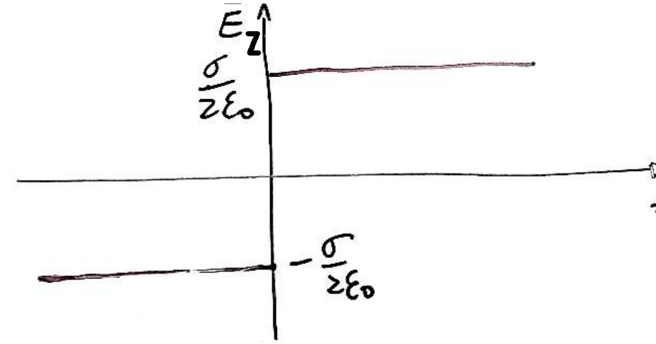
$$\bar{E}(P) = k_e \sigma z\, 2\pi \int_0^\infty \frac{r'\,dr'}{(z^2+r'^2)^{3/2}}\, \hat{e}_z$$

$$\int_0^\infty \frac{r'\,dr'}{(z^2+r'^2)^{3/2}} = -\frac{1}{(r'^2+z^2)^{1/2}}\Big|_0^\infty = 0-\left(-\frac{1}{z}\right) = \frac{1}{z}$$

$$k_e = \frac{1}{4\pi\varepsilon_0}$$

$$\boxed{\bar{E}(P) = \frac{\sigma}{2\varepsilon_0}\frac{z}{z}\hat{e}_z = \frac{\sigma}{2\varepsilon_0}\hat{e}_z} \quad z>0$$

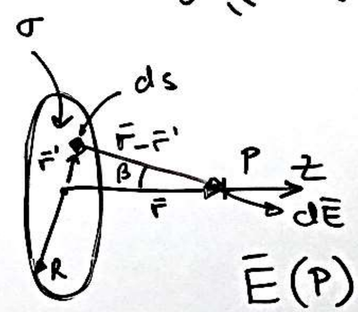
$$\boxed{\bar{E}(P) = -\frac{\sigma}{2\varepsilon_0}\hat{e}_z \quad z<0}$$



CAMPO ELECTRICO DE UN DISCO DE RADIO R

El problema es el mismo que en el caso del plano ∞, la diferencia son los límites de integración para r

plano ∞ $0 \leq r' \leq \infty \longrightarrow$ disco $0 \leq r' \leq R$

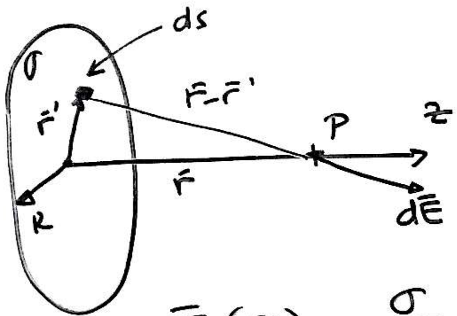


Nos interesa calcular $\bar{E}$ en el punto P sobre el eje z.

$$\bar{E}(P) = k_e \sigma \int_0^R \int_0^{2\pi} \frac{z\,r'}{(r'^2+z^2)^{3/2}}\, d\phi\, dr'\, \hat{e}_z$$

$$= k_e \sigma z\, 2\pi \left[-\frac{1}{(r'^2+z^2)^{1/2}}\right]\Bigg|_0^R = k_e \sigma 2\pi z \left[-\frac{1}{(R^2+z^2)^{1/2}} + \frac{1}{z}\right]$$

$$\boxed{\bar{E}(P) = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{(R^2+z^2)^{1/2}}\right]\hat{e}_z}$$



$$\bar{E}(P) = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{(R^2+z^2)^{1/2}}\right]\hat{e}_z$$

si $z \to 0$ $\quad \boxed{\bar{E}(z\to 0) = \frac{\sigma}{2\varepsilon_0}\,\hat{e}_z}$ idem plano ∞

si $z \to \infty$ $\quad \bar{E}(z\to\infty) = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{\left(1+\left(\frac{R}{z}\right)^2\right)^{1/2}}\right]\hat{e}_z$

$$\boxed{\bar{E}(z\to\infty) = \frac{\sigma}{2\varepsilon_0}\left[1-1\right]\hat{e}_z = \bar{0}}$$

si $z \gg R$ $\quad \bar{E}(z) = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{\left(1+\left(\frac{R}{z}\right)^2\right)^{1/2}}\right]\hat{e}_z$

usamos que $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ si $x \ll$

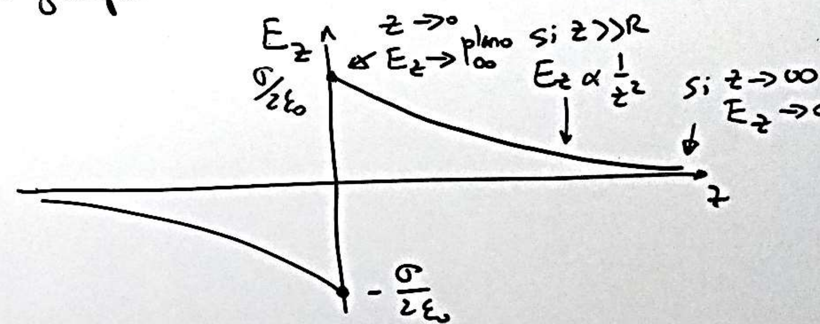
$$\bar{E}(z) = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{1+\frac{R^2}{2z^2}}\right]\hat{e}_z$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\frac{1+\frac{R^2}{2z^2}-1}{1+\frac{R^2}{2z^2}}\right)\hat{e}_z = \frac{\sigma}{2\varepsilon_0}\,\frac{R^2/2z^2}{1+\frac{R^2}{2z^2}}\,\hat{e}_z$$

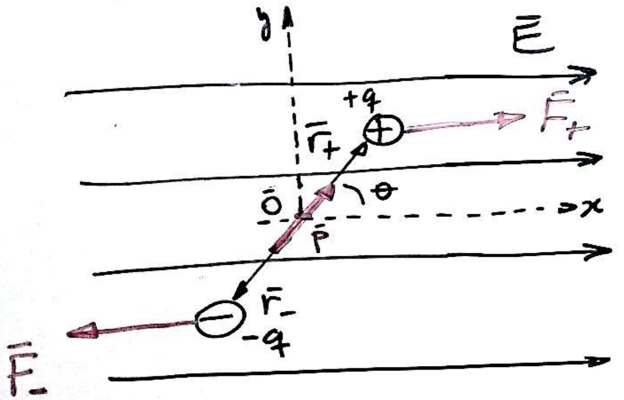
$$= \frac{\sigma}{2\varepsilon_0}\,\frac{1}{1+\frac{2z^2}{R^2}}\,\hat{e}_z \underset{\frac{2z^2}{R^2}\gg 1}{\approx} \frac{\sigma}{2\varepsilon_0}\,\frac{R^2}{2z^2}\,\hat{e}_z$$

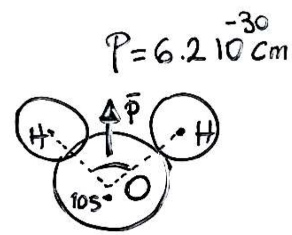
como $\sigma = \frac{Q}{\pi R^2}$ $\Rightarrow$ $\boxed{\bar{E}(z) \approx \frac{Q}{4\pi\varepsilon_0}\,\frac{1}{z^2}\,\hat{e}_z}$

Si graficamos el campo eléctrico



Dipolo en un campo eléctrico $\bar{E}$





Por un lado tenemos,

$$\bar{p} = p\,(\cos\theta\,\hat{e}_x + \text{sen}\,\theta\,\hat{e}_y)$$

si $|\bar{r}_+| = |\bar{r}_-| = a$ y $p = 2aq$

$$\bar{p} = 2aq\,(\cos\theta\,\hat{e}_x + \text{sen}\,\theta\,\hat{e}_y)$$

Por otro lado, la fuerza sobre al dipolo resulta:

$$\bar{F}_+ = q\bar{E} \quad \text{y} \quad \bar{F}_- = -q\bar{E}$$

por lo que $\bar{F} = \bar{F}_+ + \bar{F}_- = 0$

Sin enbargo existe un torque $\bar{\tau}$ respecto el origen de coordenadas $\bar{O}$

$$\bar{\tau} = \bar{r}_+ \times \bar{F}_+ + \bar{r}_- \times \bar{F}_-$$

donde

$$\bar{r}_+ = a\,(\cos\theta\,\hat{e}_x + \text{sen}\,\theta\,\hat{e}_y)$$

$$\bar{r}_- = a\,(-\cos\theta\,\hat{e}_x - \text{sen}\,\theta\,\hat{e}_y)$$

$$\bar{\tau} = a\,(\cos\theta\,\hat{e}_x + \text{sen}\,\theta\,\hat{e}_y) \times F_+\,\hat{e}_x + a\,(-\cos\theta\,\hat{e}_x - \text{sen}\,\theta\,\hat{e}_y) \times (-F_-\,\hat{e}_x)$$

$$\bar{\tau} = -a\,\text{sen}\,\theta\,F_+\,\hat{e}_z - a\,\text{sen}\,\theta\,F_-\,\hat{e}_z$$

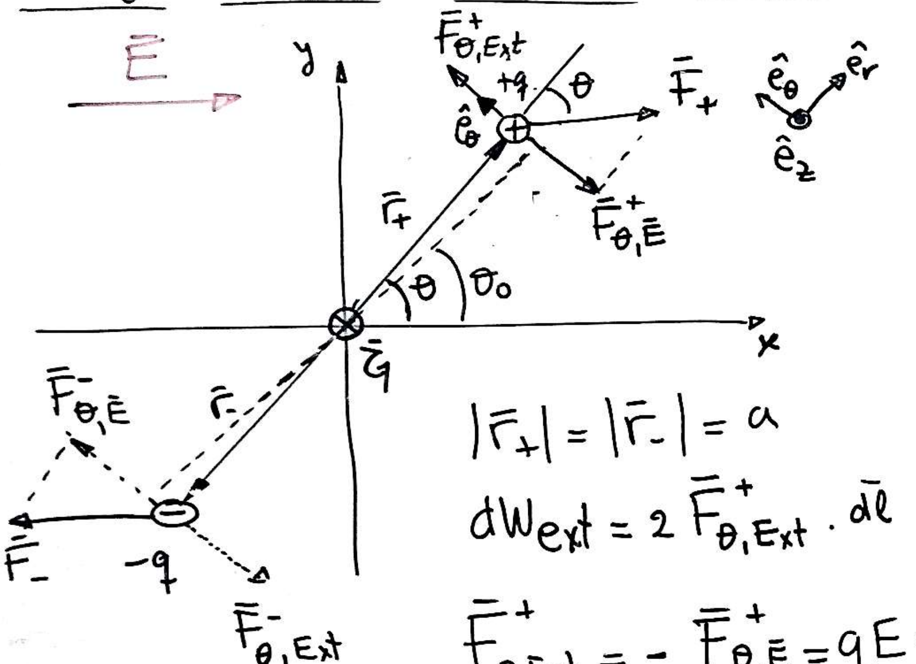
$$F_+ = F_- = F = qE$$

$$\bar{\tau} = -2a\,\text{sen}\,\theta\,F\,\hat{e}_z$$

$$\bar{\tau} = -2aq\,\text{sen}\,\theta\,E\,\hat{e}_z = -pE\,\text{sen}\,\theta\,\hat{e}_z$$

$$\boxed{\bar{\tau} = \bar{p} \times \bar{E}}$$

Energía potencial de un dipolo eléctrico



$$|\bar{r}_+| = |\bar{r}_-| = a$$

$$dW_{ext} = 2\,\bar{F}^{+}_{\theta,Ext}\cdot d\bar{l}$$

$$\bar{F}^{+}_{\theta,Ext} = -\bar{F}^{+}_{\theta,\bar{E}} = qE\,\text{sen}\theta\,\hat{e}_\theta$$

$$d\bar{l} = a\,d\theta\,\hat{e}_\theta$$

$$dW_{Ext} = 2qE\,\text{sen}\theta\;a\,d\theta$$

$$W_{Ext} = 2qaE\int_{\theta_0}^{\theta}\text{sen}\theta\,d\theta$$

$$= 2qaE\,(-\cos\theta)\Big|_{\theta_0}^{\theta} =$$

$$= -2qaE\,(\cos\theta - \cos\theta_0)$$

$$W_{EXT} = \Delta U = U_f - U_0 =$$

$$= -pE\,(\cos\theta - \cos\theta_0)$$

Si elegimos $\theta_0 = \frac{\pi}{2} \Rightarrow$

$$\Delta U = -pE\cos\theta$$

$$\boxed{\Delta U = -\bar{p}\cdot\bar{E}}$$

Como el sistema será estable en el caso de mínima energía, tenemos q' si:

- $\theta = 0$ $\quad \bar{E} \rightarrow,\ \bar{p} \rightarrow$ $\quad \Delta U = -pE$ $\quad \bar{F}_{-} \leftarrow \ominus \quad \oplus \rightarrow \bar{F}_{+}$

  sistema en equilibrio, mínima energía, estable

- $\theta = \frac{\pi}{2}$ $\quad \bar{p} \uparrow \ \rightarrow \bar{E}$ $\quad \Delta U = 0$ $\quad \bar{p}\ \oplus \rightarrow \bar{F}^{+}\ \ \rightarrow \bar{E}\ \ \leftarrow\ominus\ \bar{F}^{-}$

  sistema no equilibrado

- $\theta = \pi$ $\quad \leftarrow \bar{p} \quad \bar{E} \rightarrow$ $\quad \Delta U = pE$ $\quad \bar{F}_{+}\ \oplus \rightarrow \quad \bar{F}_{-} \leftarrow \ominus$

  Sistema en equilibrio, máxima energía, inestable