**(ii) Where is the energy stored?** Equations 2.43 and 2.45 offer two different ways of calculating the same thing. The first is an integral over the charge distribution; the second is an integral over the field. These can involve completely different regions. For instance, in the case of the spherical shell (Ex. 2.8) the charge is confined to the surface, whereas the electric field is present everywhere *outside* this surface. Where *is* the energy, then? Is it stored in the field, as Eq. 2.45 seems to suggest, or is it stored in the charge, as Eq. 2.43 implies? At the present level, this is simply an unanswerable question: I can tell you what the total energy is, and I can provide you with several different ways to compute it, but it is unnecessary to worry about *where* the energy is located. In the context of radiation theory (Chapter 11) it is useful (and in General Relativity it is *essential*) to regard the energy as being stored in the field, with a density

$$\frac{\epsilon_0}{2}E^2 = \text{energy per unit volume.} \tag{2.46}$$

But in electrostatics one could just as well say it is stored in the charge, with a density $\frac{1}{2}\rho V$. The difference is purely a matter of bookkeeping.

**(iii) The superposition principle.** Because electrostatic energy is *quadratic* in the fields, it does *not* obey a superposition principle. The energy of a compound system is *not* the sum of the energies of its parts considered separately—there are also "cross terms":

$$\begin{aligned} W_{\text{tot}} &= \frac{\epsilon_0}{2}\int E^2\,d\tau = \frac{\epsilon_0}{2}\int(\mathbf{E}_1+\mathbf{E}_2)^2\,d\tau \\ &= \frac{\epsilon_0}{2}\int(E_1^2+E_2^2+2\mathbf{E}_1\cdot\mathbf{E}_2)\,d\tau \\ &= W_1+W_2+\epsilon_0\int\mathbf{E}_1\cdot\mathbf{E}_2\,d\tau. \end{aligned} \tag{2.47}$$

For example, if you double the charge everywhere, you *quadruple* the total energy.

**Problem 2.34** Consider two concentric spherical shells, of radii $a$ and $b$. Suppose the inner one carries a charge $q$, and the outer one a charge $-q$ (both of them uniformly distributed over the surface). Calculate the energy of this configuration, (a) using Eq. 2.45, and (b) using Eq. 2.47 and the results of Ex. 2.8.

# 2.5 Conductors

## 2.5.1 Basic Properties

In an **insulator,** such as glass or rubber, each electron is attached to a particular atom. In a metallic **conductor,** by contrast, one or more electrons per atom are free to roam about at will through the material. (In liquid conductors such as salt water it is *ions* that do the moving.) A *perfect* conductor would be a material containing an *unlimited* supply of completely free charges. In real life there are no perfect conductors, but many substances come amazingly close. From this definition the basic electrostatic properties of ideal conductors immediately follow:

**(i) E = 0 inside a conductor.** Why? Because if there *were* any field, those free charges would move, and it wouldn't be electro*statics* any more. Well . . . that's hardly a satisfactory explanation; maybe all it proves is that you can't have electrostatics when conductors are present. We had better examine what happens when you put a conductor into an external electric field $\mathbf{E}_0$ (Fig. 2.42). Initially, this will drive any free positive charges to the right, and negative ones to the left. (In practice it's only the negative charges—electrons—that do the moving, but when they depart the right side is left with a net positive charge—the stationary nuclei—so it doesn't really matter which charges move; the effect is the same.) When they come to the edge of the material, the charges pile up: plus on the right side, minus on the left. Now, these **induced charges** produce a field of their own, $\mathbf{E}_1$, which, as you can see from the figure, is in the *opposite direction* to $\mathbf{E}_0$. That's the crucial point, for it means that the field of the induced charges *tends to cancel off the original field.* Charge will continue to flow until this cancellation is complete, and the resultant field inside the conductor is precisely zero.[7] The whole process is practically instantaneous.

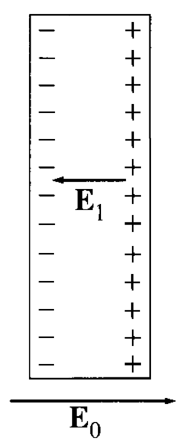


Figure 2.42

**(ii) $\rho$ = 0 inside a conductor.** This follows from Gauss's law: $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. If $\mathbf{E} = 0$, so also is $\rho$. There is still charge around, but exactly as much plus charge as minus, so the *net* charge density in the interior is zero.

**(iii) Any net charge resides on the surface.** That's the only other place it *can* be.

**(iv) A conductor is an equipotential.** For if **a** and **b** are any two points within (or at the surface of) a given conductor, $V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$, and hence $V(\mathbf{a}) = V(\mathbf{b})$.

[7] *Outside* the conductor the field is *not* zero, for here $\mathbf{E}_0$ and $\mathbf{E}_1$ do not cancel.

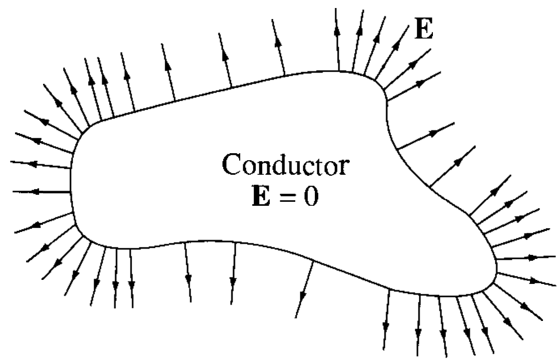


Figure 2.43

**(v) E is perpendicular to the surface, just outside a conductor.** Otherwise, as in **(i)**, charge will immediately flow around the surface until it kills off the tangential component (Fig. 2.43). (*Perpendicular* to the surface, charge cannot flow, of course, since it is confined to the conducting object.)

I think it is strange that the charge on a conductor flows to the surface. Because of their mutual repulsion, the charges naturally spread out as much as possible, but for *all* of them to go to the surface seems like a waste of the interior space. Surely we could do better, from the point of view of making each charge as far as possible from its neighbors, to sprinkle *some* of them throughout the volume... Well, it simply is not so. You do best to put *all* the charge on the surface, and this is true regardless of the size or shape of the conductor.[8]

The problem can also be phrased in terms of energy. Like any other free dynamical system, the charge on a conductor will seek the configuration that minimizes its potential energy. What property **(iii)** asserts is that the electrostatic energy of a solid object (with specified shape and total charge) is a minimum when that charge is spread over the surface. For instance, the energy of a sphere is $(1/8\pi\epsilon_0)(q^2/R)$ if the charge is uniformly distributed over the surface, as we found in Ex. 2.8, but it is greater, $(3/20\pi\epsilon_0)(q^2/R)$, if the charge is uniformly distributed throughout the volume (Prob. 2.32).

### 2.5.2 Induced Charges

If you hold a charge $+q$ near an uncharged conductor (Fig. 2.44), the two will attract one another. The reason for this is that $q$ will pull minus charges over to the near side and repel plus charges to the far side. (Another way to think of it is that the charge moves around in such a way as to cancel off the field of $q$ for points inside the conductor, where the total field must be zero.) Since the negative induced charge is closer to $q$, there is a net force of attraction. (In Chapter 3 we shall calculate this force explicitly, for the case of a spherical conductor.)

[8]By the way, the one- and two-dimensional analogs are quite different: The charge on a conducting *disk* does *not* all go to the perimeter (R. Friedberg, *Am. J. of Phys.* **61**, 1084 (1993)), nor does the charge on a conducting needle go to the ends (D. J. Griffiths and Y. Li, *Am. J. of Phys.* **64**, 706 (1996)). See Prob. 2.52.

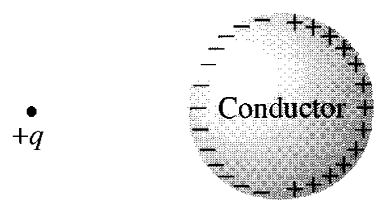


Figure 2.44

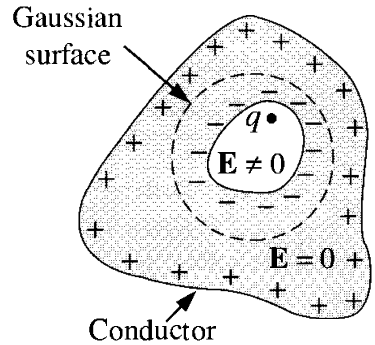


Figure 2.45

By the way, when I speak of the field, charge, or potential "inside" a conductor, I mean in the "meat" of the conductor; if there is some *cavity* in the conductor, and within that cavity there is some charge, then the field *in the cavity* will *not* be zero. But in a remarkable way the cavity and its contents are electrically isolated from the outside world by the surrounding conductor (Fig. 2.45). No external fields penetrate the conductor; they are canceled at the outer surface by the induced charge there. Similarly, the field due to charges within the cavity is killed off, for all exterior points, by the induced charge on the inner surface. (However, the compensating charge left over on the *outer* surface of the conductor effectively "communicates" the presence of $q$ to the outside world, as we shall see in Ex. 2.9.) Incidentally, the total charge induced on the cavity wall is equal and opposite to the charge inside, for if we surround the cavity with a Gaussian surface, all points of which are in the conductor (Fig. 2.45), $\oint \mathbf{E} \cdot d\mathbf{a} = 0$, and hence (by Gauss's law) the net enclosed charge must be zero. But $Q_{\text{enc}} = q + q_{\text{induced}}$, so $q_{\text{induced}} = -q$.

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**Example 2.9**

An uncharged spherical conductor centered at the origin has a cavity of some weird shape carved out of it (Fig. 2.46). Somewhere within the cavity is a charge $q$. *Question:* What is the field outside the sphere?

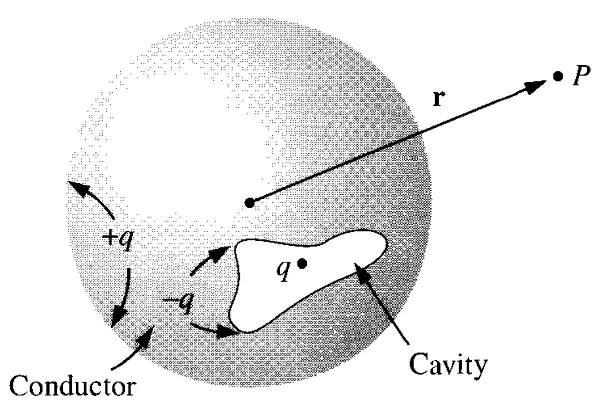


Figure 2.46

**Solution:** At first glance it would appear that the answer depends on the shape of the cavity and on the placement of the charge. But that's wrong: The answer is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

*regardless.* The conductor conceals from us all information concerning the nature of the cavity, revealing only the total charge it contains. How can this be? Well, the charge $+q$ induces an opposite charge $-q$ on the wall of the cavity, which distributes itself in such a way that its field cancels that of $q$, for all points exterior to the cavity. Since the conductor carries no net charge, this leaves $+q$ to distribute itself uniformly over the surface of the sphere. (It's *uniform* because the asymmetrical influence of the point charge $+q$ is negated by that of the induced charge $-q$ on the inner surface.) For points outside the sphere, then, the only thing that survives is the field of the leftover $+q$, uniformly distributed over the outer surface.

It may occur to you that in one respect this argument is open to challenge: There are actually *three* fields at work here, $\mathbf{E}_q$, $\mathbf{E}_{\text{induced}}$, and $\mathbf{E}_{\text{leftover}}$. All we know for certain is that the sum of the three is zero inside the conductor, yet I claimed that the first two *alone* cancel, while the third is separately zero there. Moreover, even if the first two cancel within the conductor, who is to say they still cancel for points outside? They do not, after all, cancel for points *inside* the cavity. I cannot give you a completely satisfactory answer at the moment, but this much at least is true: There *exists* a way of distributing $-q$ over the inner surface so as to cancel the field of $q$ at all exterior points. For that same cavity could have been carved out of a *huge* spherical conductor with a radius of 27 miles or light years or whatever. In that case the leftover $+q$ on the outer surface is simply too far away to produce a significant field, and the other two fields would *have* to accomplish the cancellation by themselves. So we know they *can* do it . . . but are we sure they *choose* to? Perhaps for small spheres nature prefers some complicated three-way cancellation. Nope: As we'll see in the uniqueness theorems of Chapter 3, electrostatics is very stingy with its options; there is always precisely one way—no more—of distributing the charge on a conductor so as to make the field inside zero. Having found a *possible* way, we are guaranteed that no alternative exists even in principle.

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If a cavity surrounded by conducting material is itself empty of charge, then the field within the cavity is zero. For any field line would have to begin and end on the cavity wall, going from a plus charge to a minus charge (Fig. 2.47). Letting that field line be part of a closed loop, the rest of which is entirely inside the conductor (where $\mathbf{E} = 0$), the integral

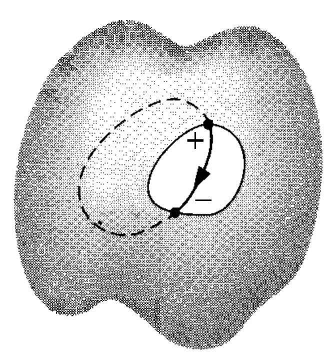


Figure 2.47

$\oint \mathbf{E} \cdot d\mathbf{l}$ is distinctly *positive*, in violation of Eq. 2.19. It follows that $\mathbf{E} = 0$ within an *empty* cavity, and there is in fact *no* charge on the surface of the cavity. (This is why you are relatively safe inside a metal car during a thunderstorm—you may get *cooked*, if lightning strikes, but you will not be *electrocuted*. The same principle applies to the placement of sensitive apparatus inside a grounded **Faraday cage,** to shield out stray electric fields. In practice, the enclosure doesn't even have to be solid conductor—chicken wire will often suffice.)

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**Problem 2.35** A metal sphere of radius $R$, carrying charge $q$, is surrounded by a thick concentric metal shell (inner radius $a$, outer radius $b$, as in Fig. 2.48). The shell carries no net charge.

(a) Find the surface charge density $\sigma$ at $R$, at $a$, and at $b$.

(b) Find the potential at the center, using infinity as the reference point.

(c) Now the outer surface is touched to a grounding wire, which lowers its potential to zero (same as at infinity). How do your answers to (a) and (b) change?

**Problem 2.36** Two spherical cavities, of radii $a$ and $b$, are hollowed out from the interior of a (neutral) conducting sphere of radius $R$ (Fig. 2.49). At the center of each cavity a point charge is placed—call these charges $q_a$ and $q_b$.

(a) Find the surface charges $\sigma_a$, $\sigma_b$, and $\sigma_R$.

(b) What is the field outside the conductor?

(c) What is the field within each cavity?

(d) What is the force on $q_a$ and $q_b$?

(e) Which of these answers would change if a third charge, $q_c$, were brought near the conductor?

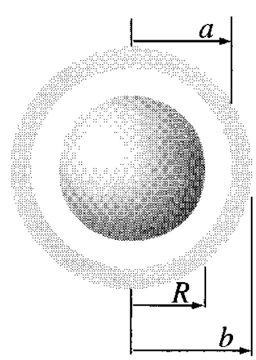


Figure 2.48

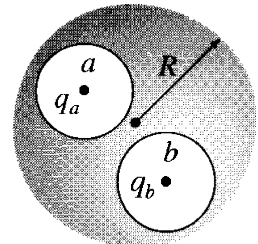


Figure 2.49