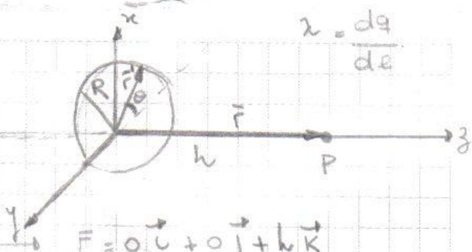
PROBLEMA 6:



$\lambda = \frac{dq}{dl}$

$$d\bar{E} = \underbrace{\frac{1}{4\pi\varepsilon_0}}_{K} \frac{dq}{|\bar r - \bar r'|^3} (\bar r - \bar r')$$

$\bar r$ = punto al que queremos evaluar el campo $\bar E$ → $\bar r = 0\vec i + 0\vec j + h\vec k$

$\bar r'$ = punto donde esta ubicada la carga eléctrica → $\bar r' = R\cos\theta\,\vec i + R\sin\theta\,\vec j + 0\vec k$

$$\begin{cases} dq = \lambda\, dl \\ dq = \lambda R\, d\theta \end{cases}$$

$$\bar r - \bar r' = -R\cos\theta\,\vec i - R\sin\theta\,\vec j + h\vec k$$

$$|\bar r - \bar r'| = (R^2\cos^2\theta + R^2\sin^2\theta + h^2)^{1/2} = (R^2+h^2)^{1/2} \to |\bar r - \bar r'|^3 = (R^2+h^2)^{3/2}$$

$$d\bar E = \frac{1}{4\pi\varepsilon}\frac{(-R\cos\theta\,\vec i - R\sin\theta\,\vec j + h\vec k)}{(R^2+h^2)^{3/2}}\lambda R\, d\theta$$

$$\bar E = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{(\overbrace{-R\cos\theta\,\vec i}^{0} - \overbrace{R\sin\theta\,\vec j}^{0} + h\vec k)}{(R^2+h^2)^{3/2}}\lambda R\, d\theta$$

$$\bar E = \frac{\lambda R}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{h}{(R^2+h^2)^{3/2}}\underbrace{d\theta}_{2\pi}\,\vec k \longrightarrow \boxed{\bar E = \frac{\lambda R h}{2\varepsilon_0}\frac{1}{(R^2+h^2)^{3/2}}\vec k}$$

(a) Ahora $\{ dq = \sigma\, da = \sigma\, r\, dr\, d\theta$

$$\bar E = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{R}\frac{h\sigma\, r\, dr\, d\theta}{(r^2+h^2)^{3/2}}\vec k = \frac{\sigma h 2\pi}{4\pi\varepsilon_0}\int_0^{R}\frac{r\, dr}{(r^2+h^2)^{3/2}}\vec k$$

$u = r^2 + h^2$
$du = 2r\, dr$

por lo tanto:

$$\int\frac{du/2}{u^{3/2}} = \cancel{\frac{1}{2}\int\frac{du}{u^{3/2}}} = \frac{1}{2}(-2)\,u^{-1/2} = -u^{-1/2}$$

$$\int_0^R\frac{r\, dr}{(r^2+h^2)^{3/2}} = -(r^2+h^2)^{-1/2} = \left.\frac{-1}{(r^2+h^2)^{1/2}}\right|_0^R = \frac{1}{(h^2)^{1/2}} - \frac{1}{(R^2+h^2)^{1/2}}$$

Quedando el campo:

$$\boxed{\bar E = \frac{\sigma h}{2\varepsilon_0}\left[\frac{1}{h} - \frac{1}{(R^2+h^2)^{1/2}}\right]\vec k}$$

(b) Analizar cuando $R \rightarrow \infty$:

$$\vec{E} = \frac{\sigma h}{2\varepsilon_0}\left[\frac{1}{h} - \frac{1}{(R^2+h^2)^{1/2}}\right]\vec{k} \quad \therefore \quad \vec{E} = \frac{\sigma h}{2\varepsilon_0 h}\vec{k} \longrightarrow \boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\vec{k}}$$

el campo de un plano infinito

Ademas analizamos $R \rightarrow 0$:

$$\vec{E} = \frac{\sigma h}{2\varepsilon_0}\frac{1}{h}\left[1 - \frac{h}{(R^2+h^2)^{1/2}}\right]\vec{k} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{h}{h(1+R^2/h^2)^{1/2}}\right]\vec{k} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{(1+R^2/h^2)^{1/2}}\right]\vec{k}$$

Utilizamos la aproximación: (para R pequeños)

$$\frac{1}{\sqrt{1-(R^2/h^2)}} \cong 1 - \frac{1}{2}\left(\frac{R^2}{h^2}\right)$$

$$\rightarrow \vec{E} \simeq \frac{\sigma}{2\varepsilon_0}\left[1 - \left(1 - \frac{1}{2}\frac{R^2}{h^2}\right)\right]\vec{k} = \frac{\sigma}{2\varepsilon_0}\frac{1}{2}\frac{R^2}{h^2}$$

y como además $dq = \sigma da$

$Q = \sigma A \rightarrow Q = \sigma \pi R^2 \rightarrow \sigma R^2 = \frac{Q}{\pi}$

por lo que:

$$\vec{E} \simeq \frac{1}{4\varepsilon_0 h^2}(\sigma R^2) = \frac{1}{4\varepsilon_0 h^2}\left(\frac{Q}{\pi}\right) \rightarrow \boxed{\vec{E} \simeq \frac{Q}{4\pi\varepsilon_0 h^2}\vec{k}}$$

el campo de una carga puntual