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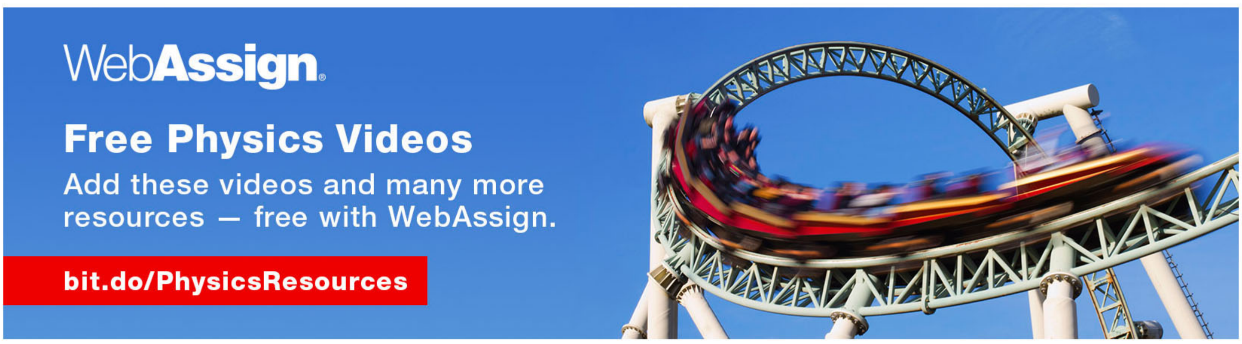

Matrix Calculus and the Stokes Parameters of Polarized Radiation

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The specification of the state of polarization of beams of radiation by Stokes parameters is discussed. Mueller's method of regarding the Stokes parameters as column matrices and optical devices as represented by 4×4 matrices is described and examples of the procedure are given. The relation of the Stokes parameters to the Poincaré sphere representation is shown.

IN recent years the use of matrix calculus for the treatment of polarized radiation has become common. The literature contains many examples of its use, but no single paper serves easily as an introduction to the subject. This paper presents a development intended for beginners and includes the relation of the Stokes parameters to the Poincaré sphere representation.

It was shown by Sir George Stokes[1] in 1852 that the most general beam of partially polarized light could be characterized by four quantities now called the "Stokes parameters." If two such general beams are superimposed incoherently the Stokes parameters are additive. Any general beam may be regarded as a superposition of an unpolarized beam and an elliptically polarized beam. The Stokes parameters have been shown[2,3] to be related to quantities which occur in the quantum-mechanical treatment of light beams.

It was pointed out by Mueller[4] on the basis of a paper by Soleillet[5] that the Stokes parameters could be regarded as the components of a column matrix or 4-vector. Thus a beam of elliptically polarized light could be represented by a 4-vector, and any change of polarization would be represented by a change in the 4-vector. Mathematically, a 4×4 matrix is an operator which changes one 4-vector into another linearly related 4-vector. Hence any optical device (Nicol prism, quarter-wave plate, etc.) which changes the state of polarization of a light beam may be represented by a 4×4 matrix. The great advantage of this representation is that the total effect of a series of optical devices traversed in turn by the beam is given by the product of the matrices for the separate devices. The difficult mathematical analysis of a series of devices is thus reduced to the routine calculation of matrix products.

STOKES PARAMETERS

Stokes showed that a general beam of light could be represented by the four parameters:

$$I=\langle E_x^2+E_y^2\rangle, \quad (1)$$

$$Q=\langle E_x^2-E_y^2\rangle, \quad (2)$$

$$U=\langle 2E_xE_y\cos\delta\rangle, \quad (3)$$

$$V=\langle 2E_xE_y\sin\delta\rangle. \quad (4)$$

The E_x and E_y are instantaneous positive definite values of the components of the electric vector on a rectangular coordinate system perpendicular to the direction of the ray and δ is the instantaneous relative phase difference restricted to the range between $-\pi$ and $+\pi$. The angular brackets represent time averages.

For an unpolarized beam the electric vector is changing rapidly and erratically with time both in amplitude and absolute phase. In this case I is the measured intensity and Q, U, V average out to zero.

For an elliptically polarized beam the electric vector is also changing rapidly, but in such a way that E_x/E_y and δ remain constant. In this case I is the measured intensity and Q, U, V will have constant values.

Clearly I measures the intensity of the beam and Q, U, V specify the state of polarization. The Stokes parameters for a few typical beams are given below. They should be written as column matrices, but to save space they will be written in a horizontal row with { } brackets to remind us they ought to be in a column. Un-

[1] G. Stokes, Trans. Cambridge Phil. Soc. 9, 399 (1852).
[2] U. Fano, J. Opt. Soc. Am. 39, 859 (1949).
[3] D. L. Falkoff and J. E. MacDonald, J. Opt. Soc. Am. 41, 861 (1951).
[4] H. Mueller, J. Opt. Soc. Am. 38, 661 (1948).
[5] P. Soleillet, Ann. phys. 12, 23 (1929).

polarized light:

$$\{E_x^2+E_y^2 \quad 0 \quad 0 \quad 0\}. \tag{5}$$

Plane polarized light:

General $\{E_x^2+E_y^2 \quad E_x^2-E_y^2 \quad 2E_xE_y \quad 0\}$, (6)

Along x axis $\{E_x^2 \quad E_x^2 \quad 0 \quad 0\}$, (7)

Along y axis $\{E_y^2 \quad -E_y^2 \quad 0 \quad 0\}$, (8)

At 45° $\{2E_x^2 \quad 0 \quad 2E_x^2 \quad 0\}$. (9)

Elliptically polarized light:

General $\{E_x^2+E_y^2 \quad E_x^2-E_y^2 \quad 2E_xE_y\cos\delta \quad 2E_xE_y\sin\delta\}$, (10)

$E_x=E_y$ $\{2E_x^2 \quad 0 \quad 2E_x^2\cos\delta \quad 2E_x^2\sin\delta\}$, (11)

Circular $\{2E_x^2 \quad 0 \quad 0 \quad 2E_x^2\}$. (12)

In a completely polarized beam $I^2=Q^2+U^2+V^2$ and the first Stokes parameter is redundant. In a mixed beam $I^2>Q^2+U^2+V^2$, the excess indicates the amount of unpolarized light present, and I is no longer redundant.

Let us consider a beam of elliptically polarized light in order to see why these particular functions serve as parameters. As shown in Fig. 1 such a beam is represented by

$$x=e_x\cos(\omega t+\epsilon_x), \tag{13}$$

$$y=e_y\cos(\omega t+\epsilon_y), \tag{14}$$

and the phase difference δ is $(\epsilon_y-\epsilon_x)$. The superposition at right angles of these two simple harmonic motions of the same frequency but out of phase by δ yields the ellipse shown. By direct (but lengthy) calculation the following relations may be proved,[6] where $\tan\beta_0=b/a$,

$$i=e_x^2+e_y^2=\text{intensity}, \tag{15}$$

$$q=e_x^2-e_y^2=(e_x^2+e_y^2)\cos 2\beta_0\cos 2\alpha_0, \tag{16}$$

$$u=2e_xe_y\cos\delta=(e_x^2+e_y^2)\cos 2\beta_0\sin 2\alpha_0, \tag{17}$$

$$v=2e_xe_y\sin\delta=(e_x^2+e_y^2)\sin 2\beta_0. \tag{18}$$

Here e_x and e_y represent peak values instead of instantaneous values as in Eq. (1) through Eq. (4) and it is unnecessary to take an average since all the factors are constant. These modified Stokes parameters i, q, u, v will differ from I, Q, U, V by a constant factor.

POINCARÉ SPHERE

The quantities q, u, v have the following geometric interpretation. Consider the vector **P** of length $(e_x^2+e_y^2)$ in Fig. 2. The vector **P** is located as shown by the azimuth angle $2\alpha_0$ and latitude angle $2\beta_0$. By ordinary trigonometry the vector **P** has the following components on the three axes:

$$P_x=(e_x^2+e_y^2)\cos 2\beta_0\cos 2\alpha_0, \tag{19}$$

$$P_y=(e_x^2+e_y^2)\cos 2\beta_0\sin 2\alpha_0, \tag{20}$$

$$P_z=(e_x^2+e_y^2)\sin 2\beta_0. \tag{21}$$

By comparing these components with Eq. (16) through Eq. (18) it is seen that $q=P_x$, $u=P_y$, $v=P_z$ and clearly

$$i^2=q^2+u^2+v^2. \tag{22}$$

Thus a beam of elliptically polarized light can be specified by the vector **P** and mapped on the sphere. This possibility was pointed out by Poincaré,[7] and has been used by numerous writers.[8–11] All plane polarizations lie in the xy

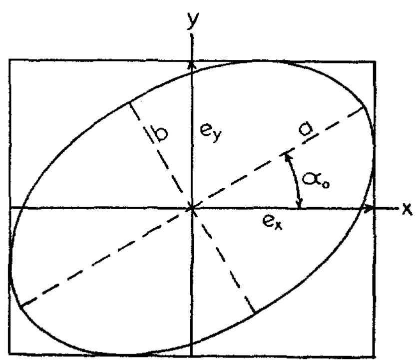

FIG. 1. Ellipse resulting from superposition at right angles of two simple harmonic motions of amplitude e having the same frequency but different phase.

[6] Max Born, *Optik* (Julius Springer, Berlin, 1933), pp. 21–23.

[7] H. Poincaré, *Traité de la Lumière* (Paris, 1892).
[8] C. A. Skinner, J. Opt. Soc. Am. **10**, 490 (1925).
[9] F. A. Wright, J. Opt. Soc. Am. **20**, 529 (1930).
[10] G. Bruhat and P. Grivet, J. phys. et radium **6**, 12 (1935).
[11] G. N. Ramachandran and S. Ramaseshan, J. Opt. Soc. Am. **42**, 49 (1952).

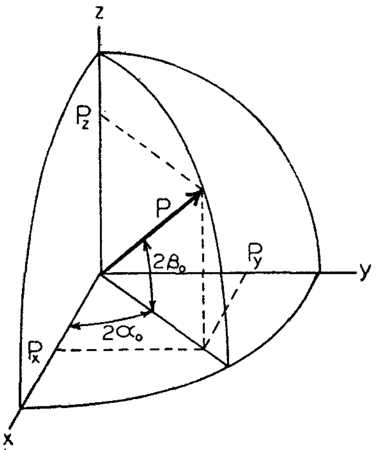

FIG. 2. Poincaré sphere representation of a beam of completely polarized light having the Stokes parameters P, P_x, P_y, and P_z.

plane, all elliptical polarizations lie above (or below) the xy plane, and circular polarization corresponds to the north (or south) pole.

The sphere can also be used to specify the setting of any compensator. Consider the following very simple compensator[2] consisting of a $\lambda/4$ plate and a plane analyzer. (See Fig. 3.) By direct computation the transmission of this compensator is

$$I/I_0=\tfrac{1}{2}(1+\cos 2\theta), \qquad (23)$$

where the factor $\cos 2\theta$ is given by

$$\cos 2\theta=\cos 2\beta \cos 2\beta_0 \cos 2(\alpha-\alpha_0) +\sin 2\beta \sin 2\beta_0. \qquad (24)$$

But the angle 2θ between $\mathbf{P}(2\beta,2\alpha)$ and $\mathbf{P}(2\beta_0, 2\alpha_0)$ is also given by Eq. (23) and Eq. (24) since

$$\cos 2\theta=l_1l_2+m_1m_2+n_1n_2, \qquad (25)$$

$$\cos 2\theta= (\cos 2\beta_0 \cos 2\alpha_0)(\cos 2\beta \cos 2\alpha) + (\cos 2\beta_0 \sin 2\alpha_0)(\cos 2\beta \sin 2\alpha) + (\sin 2\beta_0)(\sin 2\beta), \qquad (26)$$

$$\cos 2\theta=\cos 2\beta_0 \cos 2\beta \cos 2(\alpha-\alpha_0) +\sin 2\beta \sin 2\beta_0. \qquad (27)$$

Thus the transmission of any compensator for any specific polarized beam is measured by the angle 2θ between the vectors representing the beam and the compensator.

For partially polarized light we must introduce into Eq. (23) the "degree of polarization" P defined by

$$P=(I_{max}-I_{min})/(I_{max}+I_{min}). \qquad (27.1)$$

Then the transmission of a compensator will be given by

$$I/I_0=\tfrac{1}{2}(1+P\cos 2\theta), \qquad (27.2)$$

where $\cos 2\theta$ must be expressed for that particular compensator.

TRANSFORMATION MATRIX

The action of an optical device (for example, a polarizer) is to transform the polarization of a beam from one state to another. These states are described by column matrices which may be regarded as 4-vectors. A 4×4 matrix is the mathematical device for transforming one 4-vector into another to which it is linearly related. Thus we are led to the representation of any optical polarization device by a 4×4 matrix, and any series of such devices by the product of their individual matrices.

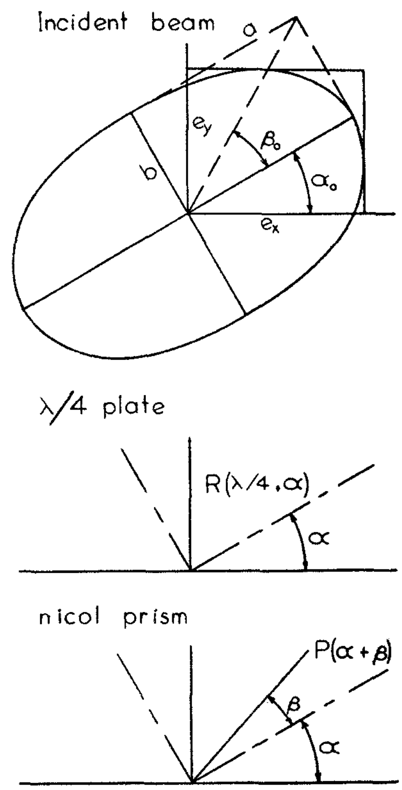

FIG. 3. Simple compensator. The incident beam falls on a quarter-wave plate followed by a Nicol prism. The quarter-wave plate is oriented at the angle α with respect to the x axis, and the Nicol prism is oriented at the angle β with respect to the quarter-wave plate.

The matrix of a partial polarizer which produces a degree of polarization P_x along the x axis and P_y along the y axis is obtained by Billings and Land.[12] It is

$$P_p=\frac{1}{2}\begin{vmatrix} P_x^2+P_y^2 & P_x^2-P_y^2 & 0 & 0 \\ P_x^2-P_y^2 & P_x^2+P_y^2 & 0 & 0 \\ 0 & 0 & 2P_xP_y & 0 \\ 0 & 0 & 0 & 2P_xP_y \end{vmatrix}. \quad (28)$$

For a perfect polarizer along the x axis, $P_x=1$, $P_y=0$, and

$$P=\frac{1}{2}\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \quad (29)$$

If P be applied as an operator (consult any treatise on matrices) to the column matrix of general elliptically polarized light, we obtain the column matrix of a beam plane polarized along the x axis,

$$P\times\begin{vmatrix} E_x^2+E_y^2 \\ E_x^2-E_y^2 \\ 2E_xE_y\cos\delta \\ 2E_xE_y\sin\delta \end{vmatrix}=2\begin{vmatrix} E_x^2 \\ E_x^2 \\ 0 \\ 0 \end{vmatrix}. \quad (30)$$

To obtain $P(\alpha)$ the matrix of a polarizer which produces perfect polarization along a line at angle α (positive counterclockwise) to the x axis we must perform the operation

$$P(\alpha)=T(-2\alpha)P(0)T(2\alpha), \quad (31)$$

where the matrix $T(2\alpha)$ is

$$T(2\alpha)=\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos2\alpha & \sin2\alpha & 0 \\ 0 & -\sin2\alpha & \cos2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \quad (32)$$

This operator rotates the x, y axis in the xy plane of the Poincaré diagram through an angle α counterclockwise from the old x axis. The result is

$$P(\alpha)=\frac{1}{2}\begin{vmatrix} 1 & \cos2\alpha & \sin2\alpha & 0 \\ \cos2\alpha & \cos^2 2\alpha & \sin2\alpha\cos2\alpha & 0 \\ \sin2\alpha & \sin2\alpha\cos2\alpha & \sin^2 2\alpha & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \quad (33)$$

For example, the application of a polarizer at 45° to a beam already polarized at 45° should produce the same beam,

$$P(45°)\times\begin{vmatrix} 2E_x^2 \\ 0 \\ 2E_x^2 \\ 0 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}\times\begin{vmatrix} 2E_x^2 \\ 0 \\ 2E_x^2 \\ 0 \end{vmatrix}=\begin{vmatrix} 2E_x^2 \\ 0 \\ 2E_x^2 \\ 0 \end{vmatrix}. \quad (34)$$

Similarly $P(-45°)$ acting on this same beam should produce zero,

$$P(-45°)\times\begin{vmatrix} 2E_x^2 \\ 0 \\ 2E_x^2 \\ 0 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}\times\begin{vmatrix} 2E_x^2 \\ 0 \\ 2E_x^2 \\ 0 \end{vmatrix}=\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}. \quad (35)$$

As an example the matrix method will be used to obtain the intensity as a function of angle for a linear analyzer applied to a general beam of elliptically polarized light. The matrix of a perfect polarizer at angle α to the x axis is $P(\alpha)$ and this multiplied into the column matrix of elliptically polarized light, Eq. (10) yields a new column matrix representing the transmitted light. However we are interested only in the intensity, so it is necessary to compute only the first element of the column, and we obtain

$$J=E_x^2\cos^2\alpha+E_y^2\sin^2\alpha +2E_xE_y\sin\alpha\cos\alpha\cos\delta. \quad (36)$$

By the procedure used for the polarizer one can obtain the matrix of a plate of retardation δ at an angle β to the x axis.

$$R(\beta,\delta)=\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\beta+\sin^2 2\beta\cos\delta & \cos2\beta\sin2\beta(1-\cos\delta) & \sin2\beta\sin\delta \\ 0 & \sin2\beta\cos2\beta(1-\cos\delta) & \sin^2 2\beta+\cos^2 2\beta\cos\delta & -\cos2\beta\sin\delta \\ 0 & -\sin2\beta\sin\delta & \cos2\beta\sin\delta & \cos\delta \end{vmatrix}. \quad (37)$$

[12] B. H. Billings and E. H. Land, J. Opt. Soc. Am. 38, 819 (1948).

The matrix of the compensator shown in Fig. (3) is then given by

$$P(\alpha+\beta)\times R(\alpha,\pi/2). \qquad (38)$$

Note that the order, from right to left, is: incident wave, first element, second element, etc.

This matrix may be used to verify Eq. (23) for the transmission of the compensator.

It is interesting to note that a half-wave plate along the x axis followed by a half-wave plate at angle β is a pure rotator of angle (-2β). Reversing the order reverses the direction of rotation.

$$\overset{R(\beta,\pi)}{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4\beta & \sin 4\beta & 0 \\ 0 & \sin 4\beta & -\cos 4\beta & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}} \times \overset{R(0,\pi)}{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}} = \overset{T(-4\beta)}{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 4\beta & -\sin 4\beta & 0 \\ 0 & \sin 4\beta & \cos 4\beta & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}.$$

Other examples of the use of this method may be found in papers by Billings.[13,14]

A method using 2×2 complex matrices has been developed by Jones, and is described in papers by Jones and Hurwitz.[15] The relationship of the Jones matrices to the Mueller matrices is discussed in reference (15). Richartz and Hsu use the Jones matrices,[16] and in the latter reference discuss their relationship to quaternions.

[13] B. H. Billings, J. Opt. Soc. Am. **41**, 966 (1951).
[14] B. H. Billings, J. Opt. Soc. Am. **42**, 12 (1952).
[15] R. C. Jones, J. Opt. Soc. Am. **31**, 488, 500 (1941); **32**, 486 (1942); **37**, 107, 110 (1947); **38**, 671 (1948); H. Hurwitz, Jr. and R. C. Jones, J. Opt. Soc. Am. **31**, 493 (1941).
[16] Hsu, Richartz, and Liang, J. Opt. Soc. Am. **37**, 99 (1947); M. Richartz and H. Y. Hsu, J. Opt. Soc. Am. **39**, 136 (1949).

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