The Description of Polarization in Classical Physics

Edward Collett

Citation: Am. J. Phys. **36**, 713 (1968); doi: 10.1119/1.1975098
View online: http://dx.doi.org/10.1119/1.1975098
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v36/i8
Published by the American Association of Physics Teachers

AMERICAN JOURNAL OF PHYSICS VOLUME 36, NUMBER 8 AUGUST 1968

The Description of Polarization in Classical Physics

EDWARD COLLETT
Atmospheric Sciences Laboratory, Fort Monmouth, New Jersey 07703

(Received 18 December 1967; revision received 10 April 1968)

An alternate derivation of the Stokes polarization parameters is presented; the parameters are obtained from the elliptic equation of polarization rather than the plane wave equations. As a result of deriving the parameters in this manner, the relationship between the polarization ellipse and the Stokes parameters is clarified. The Stokes parameters for various states of polarized light are briefly reviewed. The remainder of this article is then devoted to obtaining the Stokes parameters for a number of important physical phenomena such as the classical Zeeman effect, synchrotron radiation, Thomson scattering, reflection of electromagnetic waves by dielectric surfaces, and wave propagation in a plasma.

INTRODUCTION

The representation of plane monochromatic electromagnetic waves in the form of an ellipse to describe wave polarization is well known.[1] This description of light is very useful as it enables us to describe by means of a single equation various states of wave polarization. However, this representation is inadequate for two reasons. As the beam of light propagates through space we find that in a plane, transverse to the direction of propagation, the light vector traces out an ellipse or some special form of an ellipse such as a circle or a straight line in a time interval of the order of 10^{-15} sec. This period of time is clearly too short to allow us to follow the tracing of the ellipse. The other reason for the deficiency is that in nature the state of polarization is continually changing since the amplitudes and phases of the electromagnetic waves vary in time. Thus, the polarization ellipse is an idealization of the true behavior of radiation; it is only correct at any given instant of time. These limitations force us to consider only average values of the electromagnetic field, i.e., we must represent polarized light in terms of observables.

In order to remedy this situation, G. G. Stokes, as far back as the year 1852, introduced four quantities now known appropriately as the Stokes polarization parameters as an alternate way of describing polarization.[2] These parameters are expressed only in terms of the observables of the electromagnetic field, namely, the intensity and relative phase difference between the orthogonal wave components. In addition, they are applicable to completely polarized or partially polarized light. Thus, they give a complete description of polarized light.

In the present paper we obtain the Stokes parameters from the polarization ellipse rather than the plane wave equations as is done, for example, by Chandrasekhar.[3] Alternatively, we can obtain the Stokes parameters directly from the complex representation of the electromagnetic fields enabling us to formally bypass the time integrations. As a result, we see that the formalism of the Stokes parameters is far more versatile than originally envisioned and possess a greater usefulness than is commonly known.

I. THE STOKES PARAMETERS

We consider a pair of plane waves which are orthogonal to each other and not necessarily monochromatic to be represented by the equations[1]

$$E_x(t)=E_{0x}(t)\cos[\omega t+\delta_x(t)], \tag{1a}$$

$$E_y(t)=E_{0y}(t)\cos[\omega t+\delta_y(t)], \tag{1b}$$

where $E_{0x}(t)$ and $E_{0y}(t)$ are the instantaneous amplitudes, ω is the instantaneous angular frequency, and $\delta_x(t)$ and $\delta_y(t)$ are the instantaneous phase factors. At all times the amplitudes and phase factors fluctuate slowly compared to the rapid vibrations of the cosinusoids. The explicit removal of the term ωt between (1a) and (1b) yields the familiar polarization ellipse which is

[1] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Inc., New York, 1965), 3rd ed.

[2] G. G. Stokes, Trans. Camb. Phil. Soc., **9**, 399 (1852).

[3] S. Chandrasekhar, *Radiative Transfer* (Oxford University Press, London, 1950), p. 28.

valid, in general, only at a given instant of time,

$$\frac{E_x^2(t)}{E_{0x}^2(t)}+\frac{E_y^2(t)}{E_{0y}^2(t)}-\frac{2E_x(t)E_y(t)}{E_{0x}(t)E_{0y}(t)}\cos\delta(t)=\sin^2\delta(t), \quad (2)$$

where $\delta(t)=\delta_y(t)-\delta_x(t)$.

If we have monochromatic radiation, the amplitudes and phases are constant for all time, so (2) reduces to

$$\frac{E_x^2(t)}{E_{0x}^2}+\frac{E_y^2(t)}{E_{0y}^2}-\frac{2E_x(t)E_y(t)}{E_{0x}E_{0y}}\cos\delta=\sin^2\delta. \quad (3)$$

While E_{0x}, E_{0y}, and δ are constants, E_x and E_y continue to be implicitly dependent on time as we see from Eqs. (1a) and (1b). Hence, we have written $E_x(t)$ and $E_y(t)$ in (3). In order to represent Eq. (3) in terms of the observables of the electromagnetic field, we must take a time average over an infinite interval of time. In view of the periodicity of $E_x(t)$ and $E_y(t)$ we need average Eq. (3) only over a single period of vibration. We represent this average in time by the symbol $\langle\cdots\rangle$. We now take the time average of Eq. (3),

$$\frac{\langle E_x^2(t)\rangle}{E_{0x}^2}+\frac{\langle E_y^2(t)\rangle}{E_{0y}^2}-\frac{2\langle E_x(t)E_y(t)\rangle}{E_{0x}E_{0y}}\cos\delta=\sin^2\delta, \quad (4)$$

where

$$\langle E_i(t)E_j(t)\rangle=T^{-1}\int_0^T E_i(t)E_j(t)dt, \qquad i,j=x,y.$$

Multiplying Eq. (4) by $4E_{0x}^2E_{0y}^2$ we get

$$(4E_{0y}^2)\langle E_x^2(t)\rangle+(4E_{0x}^2)\langle E_y^2(t)\rangle-8(E_{0x}E_{0y})\langle E_x(t)E_y(t)\rangle\cos\delta=(2E_{0x}E_{0y}\sin\delta)^2. \quad (5)$$

From Eqs. (1a) and (1b), we easily find the average values indicated in Eq. (5) to be

$$\langle E_x^2(t)\rangle=\tfrac{1}{2}E_{0x}^2, \quad (6a)$$

$$\langle E_y^2(t)\rangle=\tfrac{1}{2}E_{0y}^2, \quad (6b)$$

$$\langle E_x(t)E_y(t)\rangle=\tfrac{1}{2}E_{0x}E_{0y}\cos\delta. \quad (6c)$$

Substituting Eqs. (6a), (6b), and (6c) into Eq. (5) yields

$$(2E_{0x}^2E_{0y}^2)+(2E_{0x}^2E_{0y}^2)-(2E_{0x}E_{0y}\cos\delta)^2=(2E_{0x}E_{0y}\sin\delta)^2. \quad (7)$$

Adding and subtracting the quantity $E_{0x}^4+E_{0y}^4$ to the left-hand side of Eq. (7) leads to

$$(E_{0x}^2+E_{0y}^2)^2-(E_{0x}^2-E_{0y}^2)^2-(2E_{0x}E_{0y}\cos\delta)^2=(2E_{0x}E_{0y}\sin\delta)^2. \quad (8)$$

We now write the quantities inside the parentheses as

$$s_0=E_{0x}^2+E_{0y}^2, \quad (9a)$$

$$s_1=E_{0x}^2-E_{0y}^2, \quad (9b)$$

$$s_2=2E_{0x}E_{0y}\cos\delta, \quad (9c)$$

$$s_3=2E_{0x}E_{0y}\sin\delta, \quad (9d)$$

and write Eq. (8) as

$$s_0^2=s_1^2+s_2^2+s_3^2. \quad (10)$$

The four quantities represented by Eq. (9) are the Stokes polarization parameters. We see that the Stokes parameters are simply the observable of the polarization ellipse and hence the radiation field. The first Stokes parameter is the total intensity of the radiation while the remaining quantities describe the state of polarization of the light beam.

If we now have partially polarized light, then we see that the relations given by Eq. (9) continue to be valid for very short time intervals since the amplitudes and phases fluctuate slowly. From this Chandrasekhar has shown that completely polarized or partially polarized light is represented by[3]

$$s_0^2\geq s_1^2+s_2^2+s_3^2. \quad (11)$$

The equality sign applies when we have completely polarized light. The Stokes parameters possess a number of interesting properties which are treated in an excellent manner by Chandrasekhar.[3] More modern treatments of polarization are presented by Born and Wolf,[1] McMaster,[4] and Walker.[5]

In the treatise by Born and Wolf, it is shown that the ellipticity of the polarization ellipse is given by the equation

$$\sin\chi=s_3/(s_1^2+s_2^2+s_3^2)^{1/2}, \quad (12)$$

where $\tan\chi=b/a$; b/a is the ratio of the semiminor axis to the semimajor axis. The orientation

[4] W. H. McMaster, Am. J. Phys. **22**, 351 (1954).
[5] M. J. Walker, Am. J. Phys., **22**, 170 (1954).

angle ψ of the ellipse is given by

$$\tan 2\psi = s_2/s_1, \qquad (13)$$

while the degree of polarization $\mathcal{P}$ is given by

$$\mathcal{P} = I_{\text{pol}}/I_{\text{tot}}$$
$$= (s_1^2+s_2^2+s_3^2)^{1/2}/s_0, \qquad 0 \leq \mathcal{P} \leq 1. \qquad (14)$$

The Stokes parameters can be obtained directly without having to go through the formalism of time averaging the polarization ellipse. This is accomplished by writing Eqs. (1a) and (1b) in complex notation,

$$E_x(t) = E_{0x} \exp[i(\omega t + \delta_x)]$$
$$= \mathcal{E}_{0x} \exp(i\omega t), \qquad (15a)$$

$$E_y(t) = E_{0y} \exp[i(\omega t + \delta_y)]$$
$$= \mathcal{E}_{0y} \exp(i\omega t); \qquad (15b)$$

where

$$\mathcal{E}_{0x} = E_{0x} \exp(i\delta_x), \qquad (16a)$$

and

$$\mathcal{E}_{0y} = E_{0y} \exp(i\delta_y). \qquad (16b)$$

$\mathcal{E}_{0x}$ and $\mathcal{E}_{0y}$ are complex amplitudes. Writing the plane wave equations in complex notation allows us to bypass formally the time-averaging process since the Stokes parameters are now obtained by using the formulas

$$s_0 = \mathcal{E}_{0x}\mathcal{E}_{0x}^* + \mathcal{E}_{0y}\mathcal{E}_{0y}^* = E_{0x}^2 + E_{0y}^2, \qquad (17a)$$

$$s_1 = \mathcal{E}_{0x}\mathcal{E}_{0x}^* - \mathcal{E}_{0y}\mathcal{E}_{0y}^* = E_{0x}^2 - E_{0y}^2, \qquad (17b)$$

$$s_2 = \mathcal{E}_{0x}\mathcal{E}_{0y}^* + \mathcal{E}_{0x}^*\mathcal{E}_{0y} = 2E_{0x}E_{0y}\cos\delta, \qquad (17c)$$

and

$$s_3 = i(\mathcal{E}_{0x}\mathcal{E}_{0y}^* - \mathcal{E}_{0x}^*\mathcal{E}_{0y}) = 2E_{0x}E_{0y}\sin\delta. \qquad (17d)$$

Equations (16) and (17) can be treated as the defining equations for the Stokes parameters.

As shown by Perrin, the Stokes parameters can be written as four elements of a single column matrix.[6] This is usually written as $\{s_0, s_1, s_2, s_3\}$ where the curly braces remind us that we are actually dealing with a column matrix.

In order to obtain a better understanding of the results that will be presented later, we briefly review the representations of various states of polarized light in terms of the Stokes parameters. For further details, the reader should consult the text of Shurcliff.[7]

a. Linearly Polarized Light

In this case the phase angle δ is zero or π, so the Stokes vector becomes

$$s = \{E_{0x}^2 + E_{0y}^2, E_{0x}^2 - E_{0y}^2, \pm 2E_{0x}E_{0y}, 0\}. \qquad (18)$$

It is common practice to normalize the Stokes vector, so we let

$$E_{0x}^2 + E_{0y}^2 = 1,$$

where

$$E_{0x} = \cos\alpha, \qquad E_{0y} = \sin\alpha.$$

Equation (18) then becomes

$$s = \{1, \cos 2\alpha, \sin 2\alpha, 0\}. \qquad (19)$$

Two special cases of linear polarization arise when $E_{0x}=0$ (vertical polarization) and $E_{0y}=0$ (horizontal polarization). For these states, the Stokes vectors are (in normalized form)

$$s = \{1, -1, 0, 0\} \qquad \text{(vertical polarization)}, \qquad (20a)$$

and

$$s = \{1, 1, 0, 0\} \qquad \text{(horizontal polarization)}. \qquad (20b)$$

If we have $E_{0x}=E_{0y}$ and $\delta=0$ or π, respectively, then two more states of linear polarization in the Stokes representation are

$$s = 2\{1, 0, 1, 0\} \qquad (21a)$$

$$s = 2\{1, 0, -1, 0\}. \qquad (21b)$$

b. Circular Polarization

For this state of polarization $\delta = \pi/2$ or $\delta = 3\pi/2$ and $E_{0x} = E_{0y}$. The Stokes vector is

$$s = 2\{1, 0, 0, 1\} \qquad (22a)$$

$$s = 2\{1, 0, 0, -1\}. \qquad (22b)$$

c. Elliptic Polarization

With the aid of the normalizing conditions following Eq. (18), we can write for elliptic polarization

$$s = \{1, \cos 2\alpha, \sin 2\alpha \cos\delta, \sin 2\alpha \sin\delta\}. \qquad (23)$$

Of course, we could remain with the original

[6] F. Perrin, J. Chem. Phys., **10**, 415 (1942).

[7] W. A. Shurcliff, *Polarized Light* (Harvard University Press, Cambridge, Mass., 1962).

formulation in which case we have

$$s=\{E_{0x}^2+E_{0y}^2,\ E_{0x}^2-E_{0y}^2,\ 2E_{0x}E_{0y}\cos\delta,\ 2E_{0x}E_{0y}\sin\delta\}. \quad (24)$$

We note in passing that the Stokes parameters for unpolarized light is obtained from Eq. (24) by taking the time averages of the elements, the result being

$$s=\{1, 0, 0, 0\}, \text{ normalized.} \quad (25)$$

These forms show us that if we have a radiation field which we can represent as a Stokes vector, then by comparing these new forms with those presented in this section we can immediately determine the state of polarization. In order to gain some insight into a few of the physical processes producing polarized radiation, we now discuss some useful transformation matrices.

II. TRANSFORMATION MATRICES FOR THE STOKES PARAMETERS

It is well known that when polarized light interacts with an optical device such as a wave compensator, a rotator, or a polarizer, its state of polarization is altered. Consequently, a given set of Stokes parameters must also transform to a new set of parameters when an optical device is placed in the path of a light beam. It was also shown by Perrin that if we treat the Stokes parameters as a column matrix, then a 4×4 matrix will transform the original parameters to a new set of parameters. Thus, we write

$$s'=Ts, \quad (26)$$

where s' is a 4×1 column matrix and T is a 4×4 transformation matrix representing the optical device or, for that matter, any mechanism which transforms the Stokes parameters. We now determine the transformation matrices for a compensator, a rotator, and a polarizer.

a. Compensator

A plane wave of some fixed but arbitrary polarization is incident on a wave plate with its fast axis parallel to the y axis (the phase increases by ϵ), while the slow axis is parallel to the x axis (the phase decreases by ϵ); the total phase difference is then 2ϵ. The complex field components after passing through the wave plate are

$$\mathcal{E}_x'=E_{0x}\exp[i(\delta_x-\epsilon)], \quad (27a)$$

$$\mathcal{E}_y'=E_{0y}\exp[i(\delta_y+\epsilon)], \quad (27b)$$

where the primes refer to the output complex amplitudes. According to the definitions of the Stokes parameters, we find that Eqs. (27a) and (27b) give

$$s_0'=\mathcal{E}_x'\mathcal{E}_x'^*+\mathcal{E}_y'\mathcal{E}_y'^*=E_{0x}^2+E_{0y}^2=s_0, \quad (28a)$$

$$s_1'=\mathcal{E}_x'\mathcal{E}_x'^*-\mathcal{E}_y'\mathcal{E}_y'^*=E_{0x}^2-E_{0y}^2=s_1, \quad (28b)$$

$$s_2'=\mathcal{E}_x'\mathcal{E}_y'^*+\mathcal{E}_x'^*\mathcal{E}_y' =2E_{0x}E_{0y}\cos(\delta+2\epsilon), \quad (28c)$$

and

$$s_3'=i(\mathcal{E}_x'\mathcal{E}_y'^*-\mathcal{E}_x'^*\mathcal{E}_y') =2E_{0x}E_{0y}\sin(\delta+2\epsilon). \quad (28d)$$

Equations (28c) and (28d) may be expanded and the results expressed in terms of the original Stokes parameters. Thus, we find that the transformation matrix for a wave compensator whose fast axis is parallel to the y axis is

$$T_{\text{comp}}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos2\epsilon & \sin2\epsilon \\ 0 & 0 & -\sin2\epsilon & \cos2\epsilon \end{pmatrix}. \quad (29)$$

b. Rotator

The input complex field components $\mathcal{E}_x$ and $\mathcal{E}_y$ are rotated through an angle ϑ. By the familiar rotation relations, the output complex amplitudes are then

$$\mathcal{E}_x'=E_{0x}\exp(i\delta_x)\cos\vartheta+E_{0y}\exp(i\delta_y)\sin\vartheta, \quad (30a)$$

$$\mathcal{E}_y'=-E_{0x}\exp(i\delta_x)\sin\vartheta+E_{0y}\exp(i\delta_y)\cos\vartheta. \quad (30b)$$

From the defining equations, we find

$$s_0'=\mathcal{E}_x'\mathcal{E}_x'^*+\mathcal{E}_y'\mathcal{E}_y'^*=s_0, \quad (31a)$$

$$s_1'=\mathcal{E}_x'\mathcal{E}_x'^*-\mathcal{E}_y'\mathcal{E}_y'^* =s_1\cos2\vartheta+s_2\sin2\vartheta, \quad (31b)$$

$$s_2'=\mathcal{E}_x'\mathcal{E}_y'^*+\mathcal{E}_x'^*\mathcal{E}_y' =-s_1\sin2\vartheta+s_2\cos2\vartheta, \quad (31c)$$

and

$$s_3' = i(\mathcal{E}_x'\mathcal{E}_y' - \mathcal{E}_x'^*\mathcal{E}_y') = s_3. \qquad (31d)$$

The transformation matrix for a rotator is then

$$T_{\text{rot}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\vartheta & \sin 2\vartheta & 0 \\ 0 & -\sin 2\vartheta & \cos 2\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (32)$$

c. Polarizer

For a polarizer whose axes are parallel to the x and y axes and having amplitude transmission factors P_x and P_y, respectively, the output complex amplitudes are

$$\mathcal{E}_x' = P_x E_{0x} \exp(i\delta_x), \qquad (33a)$$

$$\mathcal{E}_y' = P_y E_{0y} \exp(i\delta_y). \qquad (33b)$$

We readily find that

$$s_0' = [(P_x^2 + P_y^2)/2]s_0 + [(P_x^2 - P_y^2)/2]s_1, \qquad (34a)$$

$$s_1' = [(P_x^2 - P_y^2)/2]s_0 + [(P_x^2 + P_y^2)/2]s_1, \qquad (34b)$$

$$s_2' = P_x P_y s_2, \qquad (34c)$$

and

$$s_3' = P_x P_y s_3; \qquad (34d)$$

so, the transformation matrix for a polarizer is

$$T_{\text{pol}} = \frac{1}{2}\begin{pmatrix} P_x^2 + P_y^2 & P_x^2 - P_y^2 & 0 & 0 \\ P_x^2 - P_y^2 & P_x^2 + P_y^2 & 0 & 0 \\ 0 & 0 & 2P_xP_y & 0 \\ 0 & 0 & 0 & 2P_xP_y \end{pmatrix}. \qquad (35)$$

In order to describe the behavior of a polarizer at any angle, the matrix above must be transformed to an arbitrary coordinate axis. This is done by the transformation equation[8]

$$T_{\text{pol}}(\vartheta) = T_{\text{rot}}(-2\vartheta) T_{\text{pol}} T_{\text{rot}}(2\vartheta) \qquad (36)$$

where ϑ is defined as the angle between the polarizer transmission axis and the x axis.

[8] E. L. O'Neill, *Introduction to Statistical Optics*, (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1963) p. 139.

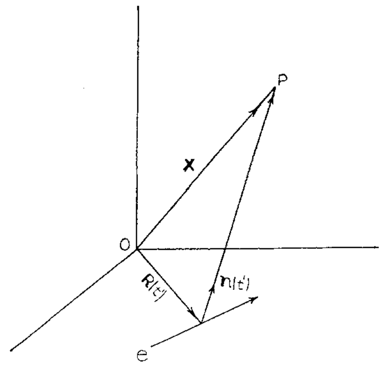

FIG. 1. Vector relations for a moving charge.

A further discussion of these results can be found throughout the cited references. With a knowledge of these matrices as references, we now obtain the Stokes parameters for some fundamental physical phenomena using complex notation for the fields.

III. THE ZEEMAN EFFECT

The fact that accelerating charges emit electromagnetic radiation is well known. The Stokes parameters afford a very convenient representation of the intensity and polarization of such radiation. As is shown by Jackson, the electric component of the radiation field due to a relativistic accelerating charge is given by[9]

$$\mathbf{E}(\mathbf{X}, t) = (e/c\kappa^3 R)[\mathbf{n} \times \{(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}]; \qquad (37)$$

where c is the speed of light, $c\boldsymbol{\beta} = \mathbf{v}$ is the velocity of the charged particle, $c\dot{\boldsymbol{\beta}} = \dot{\mathbf{v}}$ is the acceleration of the charged particle, $\kappa = (1 - \mathbf{n}\cdot\boldsymbol{\beta})$, e is the electric charge, and $\mathbf{n} = \mathbf{R}/R$ is a unit vector directed from the position of the charge to the observation point.

The relation between the vectors $\mathbf{X}$ and $\mathbf{n}$ is shown in Fig. 1. If the velocity of the charged particle is much less than the speed of light, then Eq. (37) reduces to the nonrelativistic equation

$$\mathbf{E}(\mathbf{X}, t) = (e/c^2 R)[\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{v}})]. \qquad (38)$$

We shall apply Eq. (37) in a discussion of synchrotron radiation, while Eq. (38) shall be used to discuss the classical Zeeman effect and the scattering of electromagnetic waves by charged particles. In the remainder of this section, we

[9] J. D. Jackson, *Classical Electrodynamics*, (John Wiley & Sons, New York, 1962), Chap. 14.

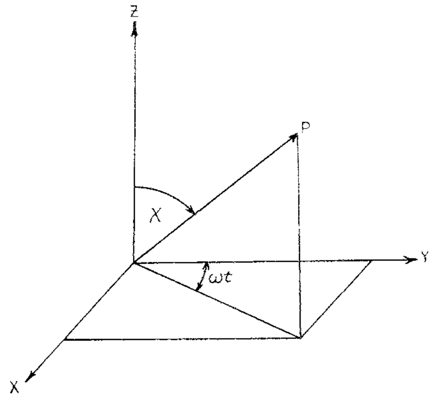

FIG. 2. The motion of an electron in a magnetic field.

determine the Stokes parameters for the classical Zeeman effect.

Consider an electron initially oscillating along the line OP (see Fig. 2) with an angular frequency ω_0 and amplitude A at an angle χ from the z axis. If a uniform constant magnetic field is parallel to the z axis ($\mathbf{H}=H\mathbf{e}_z$), the electron precesses counterclockwise as viewed along the positive z axis towards the origin with an angular frequency ω.

From Fig. 2, the resolved components of the electron position are

$$x=A\sin\chi\cos\omega_0 t\sin\omega t, \quad (39a)$$

$$y=A\sin\chi\cos\omega_0 t\cos\omega t, \quad (39b)$$

$$z=A\cos\chi\cos\omega_0 t, \quad (39c)$$

where

$$\omega=-eH/2mc.$$

By using the well-known trigonometric identities for products, we can write

$$x=(A/2)\sin\chi[\sin\omega_+ t-\sin\omega_- t], \quad (40a)$$

$$y=(A/2)\sin\chi[\cos\omega_+ t+\cos\omega_- t], \quad (40b)$$

$$z=A\cos\chi\cos\omega_0 t, \quad (40c)$$

where

$$\omega_+=\omega_0+\omega, \qquad \omega_-=\omega_0-\omega.$$

Following the familiar rule of writing Eq. (40) in complex notation, we find

$$x=[-i(A/2)\sin\chi][\exp(i\omega_+ t)-\exp(i\omega_- t)], \quad (41a)$$

$$y=[(A/2)\sin\chi][\exp(i\omega_+ t)+\exp(i\omega_- t)], \quad (41b)$$

and

$$z=(A\cos\chi)\exp(i\omega_0 t). \quad (41c)$$

Differentiation of these equations with respect to time yields

$$\ddot{x}=[i(A/2)\sin\chi][\omega_+^2\exp(i\omega_+ t)-\omega_-^2\exp(i\omega_- t)], \quad (42a)$$

$$\ddot{y}=[-(A/2)\sin\chi]\times[\omega_+^2\exp(i\omega_+ t)+\omega_-^2\exp(i\omega_- t)], \quad (42b)$$

and

$$\ddot{z}=(-A\cos\chi)\omega_0^2\exp(i\omega_0 t). \quad (42c)$$

The radiated field as given by Eq. (38) in expanded form is

$$\mathbf{E}(\mathbf{X},t)=(-e/c^2R)[\mathbf{e}_r(\mathbf{e}_r\cdot\dot{\mathbf{v}})-\dot{\mathbf{v}}], \quad (43)$$

where

$$\dot{\mathbf{v}}=\ddot{x}\mathbf{e}_x+\ddot{y}\mathbf{e}_y+\ddot{z}\mathbf{e}_z, \quad (44)$$

and where we have replaced $\mathbf{n}$ by $\mathbf{e}_r$, since the electron is oscillating along the radial direction in a spherical coordinate system. In spherical coordinates, we easily find that the term within the brackets of Eq. (43) is

$$\mathbf{e}_r(\mathbf{e}_r\cdot\dot{\mathbf{v}})-\dot{\mathbf{v}} =-\mathbf{e}_\vartheta(\ddot{x}\cos\vartheta\cos\phi+\ddot{y}\cos\vartheta\sin\phi-\ddot{z}\sin\vartheta) -\mathbf{e}_\phi(-\ddot{x}\sin\phi+\ddot{y}\cos\phi), \quad (45)$$

which shows that the radiation field is transverse. The field components are then found from Eq. (43) to be

$$\mathcal{E}_\vartheta=(e/c^2R)(\ddot{x}\cos\vartheta\cos\phi+\ddot{y}\cos\vartheta\sin\phi-\ddot{z}\sin\vartheta), \quad (46a)$$

$$\mathcal{E}_\phi=(e/c^2R)(-\ddot{x}\sin\phi+\ddot{y}\cos\phi). \quad (46b)$$

Substitution of Eq. (42) into Eq. (46) yields

$$\mathcal{E}_\vartheta=(eA/2c^2R)[i\sin\chi\cos\vartheta\{\omega_+^2\exp[i(\omega_+ t+\phi)]-\omega_-^2\exp[i(\omega_- t-\phi)]\}+2\omega_0^2\cos\chi\sin\vartheta\exp(i\omega_0 t)], \quad (47a)$$

and

$$\mathcal{E}_\phi=(eA\sin\chi/2c^2R)\{\omega_+^2\exp[i(\omega_+ t-\phi)]+\omega_-^2\exp[i(\omega_- t+\phi)]\}. \quad (47b)$$

The Stokes parameters are defined in spherical coordinates to be

$$s_0=\mathcal{E}_\vartheta\mathcal{E}_\vartheta^*+\mathcal{E}_\phi\mathcal{E}_\phi^*, \tag{48a}$$

$$s_1=\mathcal{E}_\vartheta\mathcal{E}_\vartheta^*-\mathcal{E}_\phi\mathcal{E}_\phi^*, \tag{48b}$$

$$s_2=\mathcal{E}_\vartheta\mathcal{E}_\phi^*+\mathcal{E}_\vartheta^*\mathcal{E}_\phi, \tag{48c}$$

and

$$s_3=i(\mathcal{E}_\vartheta\mathcal{E}_\phi^*-\mathcal{E}_\vartheta^*\mathcal{E}_\phi). \tag{48d}$$

We now form the appropriate products as shown by Eq. (48) using Eq. (47), drop all cross-product terms, and then average χ over a sphere of unit radius. In addition, Eq. (39) has been written such that the observation angle ϕ is zero, which means the radiation is being viewed in the x–z plane. Consequently, we must also set ϕ equal to zero in Eq. (47). We then find that the Stokes vector for the classical Zeeman effect is

$$s=\left(\frac{eA}{2c^2R}\right)^2\begin{pmatrix}\frac{2}{3}(\omega_+^4+\omega_-^4)(1+\cos^2\vartheta)+\frac{4}{3}\omega_0^4\sin^2\vartheta\\ -\frac{2}{3}(\omega_+^4+\omega_-^4)\sin^2\vartheta+\frac{4}{3}\omega_0^4\sin^2\vartheta\\ 0\\ \frac{4}{3}(\omega_+^4-\omega_-^4)\cos\vartheta\end{pmatrix}. \tag{49}$$

We now examine the Stokes vector at $\vartheta=0$, $\pi/2$, π, and $3\pi/2$. For $\vartheta=0$, we find

$$s_{\vartheta=0}=\frac{4}{3}\left(\frac{eA}{2c^2R}\right)^2\left[\omega_+^4\begin{pmatrix}1\\0\\0\\1\end{pmatrix}+\omega_-^4\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\right]. \tag{50}$$

We thus see from Eqs. (22a) and (22b) that we observe two radiating components, ω_+ and ω_-, which are right- and left-circularly polarized. At $\vartheta=\pi/2$, we have

$$s_{\vartheta=\pi/2}=\frac{2}{3}\left(\frac{eA}{2c^2R}\right)^2\begin{pmatrix}(\omega_+^4+\omega_-^4)+2\omega_0^4\\ -(\omega_+^4+\omega_-^4)+2\omega_0^4\\ 0\\ 0\end{pmatrix}$$

or

$$s_{\vartheta=\pi/2}=\frac{2}{3}\left(\frac{eA}{2c^2R}\right)^2\left[\omega_+^4\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}+\omega_-^4\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}+2\omega_0^4\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right]. \tag{51}$$

There are now three components, two of which are linearly horizontally polarized while the third is linearly vertically polarized. At $\vartheta=\pi$, we see that we obtain the same results found at $\vartheta=0$ except that the circular polarization of the components is reversed. Similarly, at $\vartheta=3\pi/2$, we obtain the same Stokes vector as Eq. (51).

Finally, the normalized intensity at $\vartheta=0$ is given by the first element of the Stokes vector

$$s_0=\omega_+^4+\omega_-^4, \qquad (52)$$

while at $\vartheta=\pi/2$ we have

$$s_0=(\omega_+^4/2)+(\omega_-^4/2)+\omega_0^4. \qquad (53)$$

These equations show us that the relative intensities of the components ω_+, ω_-, and ω_0 are 1:1:0 and $\frac{1}{2}:\frac{1}{2}:1$, respectively. As a last point, if the magnetic field is removed, then $H=0$ and $\omega_+=\omega_-=\omega_0$, and Eq. (49) is easily shown to reduce to the Stokes vector for unpolarized light.

Other useful facts about the polarization such as the ellipticity, orientation angle, and the degree of polarization are readily obtained from Eqs. (12), (13), and (14), respectively.

IV. SYNCHROTRON RADIATION

The radiation emitted from highly relativistic charges is known as synchrotron radiation after its discovery in the operation of the synchrotron. The electric field component due to the relativistically moving charge is given by Eq. (37),

$$\mathbf{E}(\mathbf{X},t)=(e/c\kappa^3R)[\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}]. \qquad (37)$$

The motion of a charged particle in a uniform and constant magnetic field is illustrated in Fig. 3.

From the equation of motion, the particle velocity is found to be (assuming the velocity of the particle along the z axis is negligible)[10]

$$\mathbf{v}=-a\omega(\sin\omega t\mathbf{e}_x-\cos\omega t\mathbf{e}_y), \qquad (54)$$

where

$$\omega=(eH/mc)(1-v^2c^2)^{1/2}.$$

In complex notation, Eq. (54) becomes

$$\mathbf{v}=a\omega[i\mathbf{e}_x+\mathbf{e}_y]\exp(i\omega t), \qquad (55)$$

so the acceleration is

$$\dot{\mathbf{v}}=a\omega^2[-\mathbf{e}_x+i\mathbf{e}_y]\exp(i\omega t). \qquad (56)$$

The amplitudes of the complex velocity and acceleration in spherical coordinates are then found to be

$$\mathbf{v}=ia\omega e^{-i\phi}[\sin\vartheta\mathbf{e}_r+\cos\vartheta\mathbf{e}_\vartheta-i\mathbf{e}_\phi], \qquad (57)$$

$$\dot{\mathbf{v}}=-a\omega^2e^{-i\phi}[\sin\vartheta\mathbf{e}_r+\cos\vartheta\mathbf{e}_\vartheta-i\mathbf{e}_\phi]. \qquad (58)$$

If we are far from the source of radiation, which is the usual condition, then from Fig. 3 we see that

$$\mathbf{n}\simeq\mathbf{X}/R=\mathbf{e}_r. \qquad (59)$$

With this simplification, we now write Eq. (37) in complex notation as

$$\mathbf{E}(\mathbf{X},t)=(e/c^2\kappa^3R)\{[\mathbf{e}_r\times(\mathbf{e}_r\times\dot{\mathbf{v}})]-[\mathbf{e}_r\times((\mathbf{v}/c)\times\dot{\mathbf{v}}^*)]\}. \qquad (60)$$

We have written the acceleration in the second term of Eq. (60) as the complex conjugate according to the usual rule when we vector multiply complex quantities. The evaluation of the terms within the curly braces of Eq. (60) leads to the complex components of the field,

$$\mathcal{E}_\vartheta=(e/c^2\kappa^3R)(a\omega^2e^{-i\phi}\cos\vartheta). \qquad (61a)$$

$$\mathcal{E}_\phi=(-e/c^2\kappa^3R)[ia\omega^2e^{-i\phi}-(a^2\omega^3/c)\sin\vartheta]. \qquad (61b)$$

Since the Stokes parameters represent the energy per unit area detected at an observation point at a time t due to radiation emitted by the charged particle at a time $t'=t-R(t')/c$, we must transform the Stokes parameters from s_0 to s_0dt/dt', etc. The effect of this is to simply change κ^6 to κ^5 in the denominator of the final expression given below. Forming the products as given by Eq. (48), we find the relativistic Stokes vector after setting ϕ to zero as explained earlier

$$s=\frac{e^2\beta^4}{a^2(1-\beta\cos\vartheta)^5R^2}\left[\begin{pmatrix}1+\cos^2\vartheta\\ -\sin^2\vartheta\\ 0\\ -2\cos\vartheta\end{pmatrix}+\beta^2\begin{pmatrix}\sin^2\vartheta\\ \sin^2\vartheta\\ \sin2\vartheta/\beta\\ 0\end{pmatrix}\right]. \qquad (62)$$

[10] G. Bekefi, *Radiation Processes in Plasmas*, (John Wiley & Sons, Inc., New York, 1966), p. 177.

Comparing this result with Eq. (23), we see that the emitted radiation is elliptically polarized.

For the special case of $\vartheta=0$, the Stokes vector for a relativistically moving charge is

$$s_{\vartheta=0}=\frac{2e^2\beta^4}{a^2(1-\beta)^5R^2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \quad (63)$$

which shows that the radiation is left circularly polarized. For $\vartheta=\pi/2$ we find that the Stokes vector is

$$s_{\vartheta=\pi/2}=\frac{e^2a^2\omega^4}{c^4R^2}\left[\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}+\beta^2\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right], \quad (64)$$

where $\beta=v/c=a\omega/c$. At this position in the field, the radiation is observed to have a vertically polarized component and a relativistically horizontally polarized component. At $\vartheta=\pi$, we see that the radiation according to Eq. (22a) is right circularly polarized.

For $\beta\ll1$, the nonrelativistic regime, Eq. (62) reduces to

$$s=\frac{e^2a^2\omega^4}{c^4R^2}\begin{pmatrix}1+\cos^2\vartheta\\-\sin^2\vartheta\\0\\-2\cos\vartheta\end{pmatrix}, \quad (65)$$

where $\omega=eH/mc$. This is the Stokes vector for a charge rotating in a circle in the x–y plane.

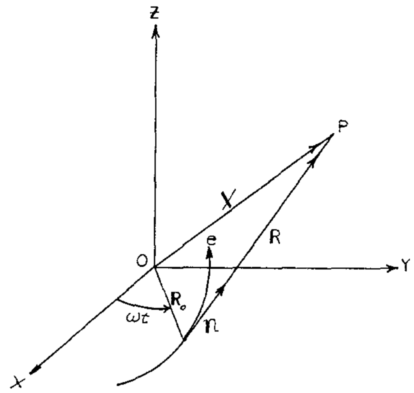

FIG. 3. Path of a charged particle in a magnetic field.

V. THOMSON SCATTERING

We now determine the Stokes parameters for the scattering of electromagnetic waves by an electron located at the origin of a Cartesian coordinate system. This is illustrated in Fig. 4. The incident electric field is transverse and propagating along the z axis and is represented by the equation

$$\mathbf{E}(\mathbf{X},t)=\mathbf{E}(\mathbf{X})e^{i\omega t},$$

where

$$\mathbf{E}(\mathbf{X})=E_{0x}\exp(i\delta_x)\mathbf{e}_x+E_{0y}\exp(i\delta_y)\mathbf{e}_y. \quad (66)$$

The motion of the electron obeys the equation

$$\dot{\mathbf{v}}=-(e/m)\mathbf{E}$$
$$=-(e/m)[E_{0x}\exp(i\delta_x)\mathbf{e}_x+E_{0y}\exp(i\delta_y)\mathbf{e}_y]e^{i\omega t}; \quad (67)$$

so, the complex acceleration amplitude is

$$\dot{\mathbf{v}}=-(e/m)[E_{0x}\exp(i\delta_x)\mathbf{e}_x+E_{0y}\exp(i\delta_y)\mathbf{e}_y]. \quad (68)$$

Transforming Eq. (68) to spherical coordinates and substituting the result into Eq. (43), we find the components of the field are transverse and are given by

$$\mathcal{E}_\vartheta=(-e/mc^2R)[E_{0x}\exp(i\delta_x)\cos\vartheta\cos\phi+E_{0y}\exp(i\delta_y)\cos\vartheta\sin\phi], \quad (69a)$$

and

$$\mathcal{E}_\phi=(-e/mc^2R)[-E_{0x}\exp(i\delta_x)\sin\phi+E_{0y}\exp(i\delta_y)\cos\phi]. \quad (69b)$$

The Stokes vector for Thomson scattering is then easily found from Eqs. (48) and (69) to be

$$s=\frac{e^4}{m^2c^4R^2}\begin{pmatrix}s_0(1+\cos^2\vartheta)-s_1\sin^2\vartheta\cos2\phi-s_2\sin^2\vartheta\sin2\phi\\-s_0\sin^2\vartheta+s_1(1+\cos^2\vartheta)\cos2\phi-s_2(1+\cos^2\vartheta)\sin2\phi\\-2s_1\cos\vartheta\sin2\phi+2s_2\cos\vartheta\cos2\phi\\2s_3\cos\vartheta\end{pmatrix} \quad (70)$$

In expressing this result, we have replaced the terms involving the amplitudes and phase of the incident wave by the incident Stokes parameters. Since we have incident radiation which is reradiated by the electron, we can write the transformation matrix. In addition, the angle ϕ is arbitrary and can be set equal to zero. The transformation matrix is seen from Eq. (70) to be

$$T_{\mathrm{scat}}=\frac{e^4}{2m^2c^4R^2}\begin{pmatrix}1+\cos^2\vartheta & -\sin^2\vartheta & 0 & 0\\ -\sin^2\vartheta & (1+\cos^2\vartheta) & 0 & 0\\ 0 & 0 & 2\cos\vartheta & 0\\ 0 & 0 & 0 & 2\cos\vartheta\end{pmatrix}, \qquad (71)$$

which we see from Eq. (35) is the equation of a polarizer.

If the incident radiation is initially unpolarized, then the initial Stokes vector is $\{s_0, 0, 0, 0\}$, and we find the scattered Stokes parameters to be

$$s=\frac{1}{2}\left(\frac{e^2}{mc^2R}\right)^2\begin{pmatrix}(1+\cos^2\vartheta)s_0\\ -\sin^2\vartheta s_0\\ 0\\ 0\end{pmatrix}. \qquad (72)$$

This result states that the initial unpolarized radiation becomes linearly polarized upon being scattered by the electron.

The scattering cross section is defined to be

$$\frac{d\sigma}{d\Omega}=\frac{\text{energy radiated/unit time/unit solid angle}}{\text{incident energy/unit area/unit time}}. \qquad (73)$$

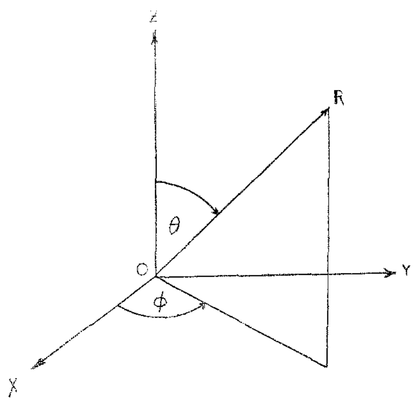

FIG. 4. Scattering of electromagnetic waves by an electron.

From Eq. (72), we see that the ratio of the scattered to incident Stokes parameters is the differential cross section

$$d\sigma/d\Omega=(e/mc^2)^2\tfrac{1}{2}(1+\cos^2\vartheta). \qquad (74)$$

Equation (74) is the well-known Thomson formula for the scattering of radiation by a free charge and is appropriate for the scattering of X rays by electrons or gamma rays by protons. The degree of polarization of the scattered radiation is found from Eq. (14) to be

$$\mathcal{P}=[\sin^2\vartheta/(1+\cos^2\vartheta)]. \qquad (75)$$

VI. REFLECTION OF ELECTROMAGNETIC WAVES BY DIELECTRIC SURFACES

As another example of the use of the Stokes parameters to describe physical phenomena in classical physics, we now consider the polarization state of a reflected electromagnetic wave at a dielectric boundary. The electric fields at the boundary separating two media with permittivity and permeability constants ϵ_1, μ_1, and ϵ_2, μ_2, respectively, is given by Fresnel's equations (in complex notation)[11]

$$\mathcal{E}_{m\perp}=\frac{\mu_2\cos i-\eta_{21}\mu_1\cos r}{\mu_2\cos i+\eta_{21}\mu_1\cos r}E_\perp\exp(i\delta_\perp), \qquad (76a)$$

$$\mathcal{E}_{b\perp}=\frac{2\mu_2\cos i}{\mu_2\cos i+\eta_{21}\mu_1\cos r}E_\perp\exp(i\delta_\perp), \qquad (76b)$$

$$\mathcal{E}_{m\parallel}=\frac{\eta_{21}\mu_1\cos i-\mu_2\cos r}{\eta_{21}\mu_1\cos i+\mu_2\cos r}E_\parallel\exp(i\delta_\parallel), \qquad (76c)$$

[11] N. Tralli, *Classical Electromagnetic Theory*, McGraw-Hill Book Co., New York, 1963), Chap. 11.

and

$$\mathcal{E}_{b||} = \frac{2\mu_2 \cos i}{\eta_{21}\mu_1 \cos i + \mu_2 \cos r} E_{||} \exp(i\delta_{||}), \quad (76d)$$

where $\eta_{21} = \sin i/\sin r$. The subscripts m and b refer to the reflected and refracted wave while the symbols $||$ and $\perp$ are the components in the plane of the paper and perpendicular to the paper. This is illustrated by Fig. 5. For a dielectric, $\mu_1 = \mu_2 = 1$ so that Eq. (76) reduces to

$$\mathcal{E}_{m\perp} = [-\sin(i-r)/\sin(i+r)] E_\perp \exp(i\delta_\perp), \quad (77a)$$

$$\mathcal{E}_{b\perp} = [2\sin r \cos i/\sin(i+r)] E_\perp \exp(i\delta_\perp), \quad (77b)$$

$$\mathcal{E}_{m||} = [\tan(i-r)/\tan(i+r)] E_{||} \exp(i\delta_{||}), \quad (77c)$$

and

$$\mathcal{E}_{b||} = [2\sin r \cos i/\sin(i+r)\cos(i-r)] E_{||} \exp(i\delta_{||}). \quad (77d)$$

The incident Stokes parameters are defined to be

$$s_0 = \mathcal{E}_\perp \mathcal{E}_\perp^* + \mathcal{E}_{||}\mathcal{E}_{||}^* = E_\perp^2 + E_{||}^2, \quad (78a)$$

$$s_1 = \mathcal{E}_\perp \mathcal{E}_\perp^* - \mathcal{E}_{||}\mathcal{E}_{||}^* = E_\perp^2 - E_{||}^2, \quad (78b)$$

$$s_2 = \mathcal{E}_\perp \mathcal{E}_{||}^* + \mathcal{E}_\perp^* \mathcal{E}_{||} = 2E_\perp E_{||}\cos\delta, \quad (78c)$$

and

$$s_3 = i(\mathcal{E}_\perp \mathcal{E}_{||}^* - \mathcal{E}_\perp^* \mathcal{E}_{||}) = 2E_\perp E_{||}\sin\delta, \quad (78d)$$

where $\delta = \delta_\perp - \delta_{||}$. The Stokes parameters for reflections are

$$s_0 = \mathcal{E}_{m\perp}\mathcal{E}_{m\perp}^* + \mathcal{E}_{m||}\mathcal{E}_{m||}^* = \left[\frac{\sin(i-r)}{\sin(i+r)}\right]^2 E_\perp^2 + \left[\frac{\tan(i-r)}{\tan(i+r)}\right]^2 E_{||}^2, \quad (79a)$$

$$s_1 = \mathcal{E}_{m\perp}\mathcal{E}_{m\perp}^* - \mathcal{E}_{m||}\mathcal{E}_{m||}^* = \left[\frac{\sin(i-r)}{\sin(i+r)}\right]^2 E_\perp^2 - \left[\frac{\tan(i-r)}{\tan(i+r)}\right]^2 E_{||}^2, \quad (79b)$$

$$s_2 = \mathcal{E}_{m\perp}\mathcal{E}_{m||}^* + \mathcal{E}_{m\perp}^*\mathcal{E}_{m||} = +2\left[\frac{\sin^2(i-r)\cos(i+r)}{\cos(i-r)\sin^2(i+r)}\right] E_\perp E_{||}\cos\delta, \quad (79c)$$

and

$$s_3 = i(\mathcal{E}_{m\perp}\mathcal{E}_{m||}^* - \mathcal{E}_{m\perp}^*\mathcal{E}_{m||}) = +2\left[\frac{\sin^2(i-r)\cos(i+r)}{\cos(i-r)\sin^2(i+r)}\right] E_\perp E_{||}\sin\delta. \quad (79d)$$

Expressing Eq. (79) only in terms of the incident Stokes parameters by using Eq. (78), the transformation matrix for reflection from a dielectric surface is found to have the form

$$T_{\text{pol}} = \frac{1}{2}\begin{pmatrix} \left[\frac{\sin^2(i-r)}{\sin^2(i+r)} + \frac{\tan^2(i-r)}{\tan^2(i+r)}\right] & \left[\frac{\sin^2(i-r)}{\sin^2(i+r)} - \frac{\tan^2(i-r)}{\tan^2(i+r)}\right] & 0 & 0 \\ \left[\frac{\sin^2(i-r)}{\sin^2(i+r)} - \frac{\tan^2(i-r)}{\tan^2(i+r)}\right] & \left[\frac{\sin^2(i-r)}{\sin^2(i+r)} + \frac{\tan^2(i-r)}{\tan^2(i+r)}\right] & 0 & 0 \\ 0 & 0 & \frac{-2\sin^2(i-r)\cos(i+r)}{\cos(i-r)\sin^2(i+r)} & 0 \\ 0 & 0 & 0 & \frac{-2\sin^2(i-r)\cos(i+r)}{\cos(i-r)\sin^2(i+r)} \end{pmatrix}. \quad (80)$$

If we now compare the matrix of Eq. (80) with Eq. (35), we see that a dielectric, e.g., glass, which reflects electromagnetic radiation behaves as a polarizer; the form of Eq. (80) is the analog of a polarizer. We also see that if the incident radiation is unpolarized so the normalized Stokes vector is {1, 0, 0, 0}, the resultant vector is linearly polarized, while the degree of polarization

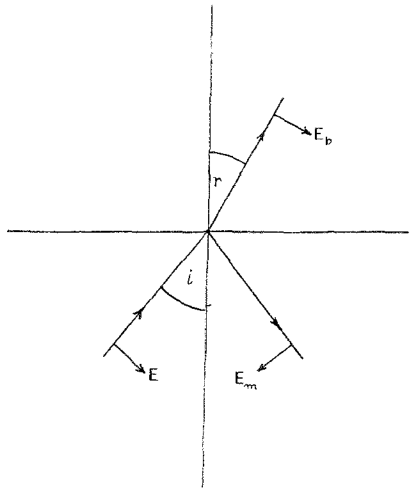

FIG. 5. Reflection and refraction of a plane wave.

is found to be

$$\mathcal{P} = \sin^2(i-r)/[1+\cos^2(i-r)]. \quad (81)$$

When $i+r=\pi/2$ (the Brewster angle), the transformation matrix of Eq. (80) becomes

$$T=\tfrac{1}{2}\cos^2 2i \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (82)$$

If the initial radiation is unpolarized, the reflected Stokes vector is found to be proportional to $\{1, 1, 0, 0\}$ and the reflected radiation is completely horizontally polarized according to Eq. (20b). In other words, there is no longer any vertical component of the field in the reflected wave.

If the angle of incidence is greater than the critical angle, we have total reflection of the incident wave. In this case, Fresnel's equations reduce tc

$$\mathcal{E}_{m\perp} = E_\perp \exp[i(\epsilon_\perp+\delta_\perp)], \quad (83a)$$

where

$$\tan(\epsilon_\perp/2) = (\eta^2 \sin^2 i - 1)^{1/2}/\eta \cos i,$$

and η is the index of refraction of the denser medium in which the incident wave is propagating. Similarly, the parallel field component is found to be

$$\mathcal{E}_{m\|} = E_\| \exp[i(\epsilon_\| + \delta_\|)], \quad (83b)$$

where

$$\tan(\epsilon_\|/2) = \eta(\eta^2 \sin^2 i - 1)^{1/2}/\cos i.$$

The Stokes parameters for the case where we have total reflection are easily found to be

$$s_0 = \mathcal{E}_{m\perp}\mathcal{E}_{m\perp}^* + \mathcal{E}_{m\|}\mathcal{E}_{m\|}^*$$
$$= E_\perp^2 + E_\|^2 \quad (84a)$$
$$s_1 = \mathcal{E}_{m\perp}\mathcal{E}_{m\perp}^* - \mathcal{E}_{m\|}\mathcal{E}_{m\|}^*$$
$$= E_\perp^2 - E_\|^2 \quad (84b)$$
$$s_2 = \mathcal{E}_{m\perp}\mathcal{E}_{m\|}^* + \mathcal{E}_{m\perp}^*\mathcal{E}_{m\|}$$
$$= 2E_\perp E_\| \cos(\delta+\epsilon), \quad (84c)$$

and

$$s_3 = i(\mathcal{E}_{m\perp}\mathcal{E}_{m\|}^* - \mathcal{E}_{m\perp}^*\mathcal{E}_{m\|})$$
$$= 2E_\perp E_\| \sin(\delta+\epsilon); \quad (84d)$$

where

$$\delta = \delta_\perp - \delta_\| \quad \text{and} \quad \epsilon = \epsilon_\perp - \epsilon_\|.$$

The transformation matrix for total reflection after expanding Eqs. (84c) and (84d) is

$$T_{\text{tot refl}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\epsilon & \sin\epsilon \\ 0 & 0 & -\sin\epsilon & \cos\epsilon \end{pmatrix}. \quad (85)$$

Comparing Eq. (85) with Eq. (29), we see that a dielectric in which there is total reflection can be used as a wave compensator.

As a final point, if the incident wave is initially linearly polarized the Stokes vector is given by (19)

$$s = \{1, \cos 2\alpha, \sin 2\alpha, 0\}.$$

The resultant reflected Stokes vector is found by matrix multiplication of Eqs. (85) and (19) and the result is

$$s = \{1, \cos 2\alpha, \sin 2\alpha \cos\epsilon, \sin 2\alpha \sin\epsilon\}. \quad (86)$$

Thus for total reflection the linearly polarized light, which is incident on the dielectric, becomes elliptically polarized on reflection.

VII. WAVE PROPAGATION IN A PLASMA

As a final example of the use of the Stokes parameters to describe radiation phenomena in classical physics, we now consider the polarization state of a wave propagating in a plasma. In addition, there is a uniform and constant magnetic field present. If the direction of propagation of the electromagnetic wave is parallel to the magnetic field, then one can show that the wave equation becomes[11]

$$\nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}-\frac{\omega_p^2}{c^2}\frac{1}{(1\pm\omega_g/\omega)}\mathbf{E}=0, \quad (87)$$

where

$$\omega_p^2=\mu_0\eta e^2c^2/m, \qquad \omega_g=eB/m.$$

The solution of Eq. (87) is

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\exp[-i(\omega t-\boldsymbol{\kappa}_\pm\cdot\mathbf{r})], \quad (88)$$

where

$$\boldsymbol{\kappa}_\pm^2=\frac{\omega^2}{c^2}\left(1-\frac{\omega_p^2}{\omega^2}\frac{1}{1\pm\omega_g/\omega}\right).$$

Now the interesting case arises when the incident waves after passing through the plasma are represented by

$$E_x(\mathbf{r},t)=E_{0x}\exp[-i(\omega t-\boldsymbol{\kappa}_+\cdot\mathbf{r})+i\delta_x], \quad (89a)$$

$$E_y(\mathbf{r},t)=E_{0y}\exp[-i(\omega t-\boldsymbol{\kappa}_-\cdot\mathbf{r})+i\delta_y]. \quad (89b)$$

In complex notation, these field amplitudes are

$$\mathcal{E}_x=E_{0x}\exp(i\boldsymbol{\kappa}_+\cdot\mathbf{r}+i\delta_x), \quad (90a)$$

and

$$\mathcal{E}_y=E_{0y}\exp(i\boldsymbol{\kappa}_-\cdot\mathbf{r}+i\delta_y). \quad (90b)$$

Forming the Stokes parameters in the usual way, we find

$$s_0'=s_0, \quad (91a)$$

$$s_1'=s_1, \quad (91b)$$

$$s_2'=2E_{0x}E_{0y}\cos(\boldsymbol{\kappa}\cdot\mathbf{r}+\delta), \quad (91c)$$

and

$$s_3'=2E_{0x}E_{0y}\sin(\boldsymbol{\kappa}\cdot\mathbf{r}+\delta); \quad (91d)$$

where $\boldsymbol{\kappa}=\boldsymbol{\kappa}_--\boldsymbol{\kappa}_+$ and $\delta=\delta_y-\delta_x$. The transformation matrix for the plasma is then seen to be

$$T_{\text{plasma}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\boldsymbol{\kappa}\cdot\mathbf{r}&\sin\boldsymbol{\kappa}\cdot\mathbf{r}\\0&0&-\sin\boldsymbol{\kappa}\cdot\mathbf{r}&\cos\boldsymbol{\kappa}\cdot\mathbf{r}\end{pmatrix}. \quad (92)$$

Thus, a plasma that has a magnetic field present behaves as a wave compensator.

VIII. CONCLUSIONS

We have shown that the Stokes polarization parameters provide a simple and elegant formulation of the description of electromagnetic radiation in classical electrodynamics. The Stokes parameters are also known to be directly related to the elements of the density matrix in quantum mechanics. Consequently, they are part of the bridge between classical and quantum electrodynamics and should be part of every modern treatment of optics and electromagnetic theory.

ACKNOWLEDGMENTS

The author wishes to thank Dr. P. M. Livingston for introducing and subsequently discussing the Stokes parameters with me. In addition, I wish to express my deep appreciation to K. M. Barnett for his support throughout this work.